Class of 1880 Exam

April 19, 2016

Solve the problems in the space provided. Show your work and explain your answers as completely as possible. If you run out of room for an answer, continue on the back of the page.

Time available: 2 hours!

Full name and e-mail: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Find all triples of integers a, b, c, such that

$$a^2 + b^2 = 3c^2.$$

2. The numbers $1, 2, \ldots, 64$ are written in the squares of a 8×8 grid, such that each square contains exactly one number, and each number from $1, \ldots, 64$ appears exactly once.

   Define the **gap** of a table to be the maximal difference between two numbers in adjacent squares (squares which have a common side or vertex). For example the adjacent squares of x are the following:

	x	

   (a) (5 points) Show that for every such table the gap is at least 9.

   (b) (5 points) Find a table with gap exactly 9.

3. (10 points) Let $n \geq 1$ be an integer and let $a_1, a_2, \ldots, a_n > 0$ be n positive real numbers. Let

$$p(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_{n-1} x - a_n$$

be a polynomial of degree n. Show that $p(x)$ has exactly one positive real root.

4. Let s, t be two positive real numbers. Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ be 8 points on a circle in that order and P be the convex octagon with these vertices.

(a) (5 points) Suppose that the sides of P have lengths s and t as follows: $|A_1A_2| = |A_3A_4| = |A_5A_6| = |A_7A_8| = t$ and $|A_2A_3| = |A_4A_5| = |A_6A_7| = |A_8A_1| = s$. What is the area of P (in terms of s and t)?

(b) (5 points) Suppose that the sides of P have lengths s and t as follows: $|A_1A_2| = |A_2A_3| = |A_3A_4| = |A_4A_5| = t$ and $|A_5A_6| = |A_6A_7| = |A_7A_8| = |A_8A_1| = s$. What is the area of P (in terms of s and t)?