Preliminary Examination, Part I

Tuesday morning, August 30, 2005

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $T(x, y, z, w) = (a, b, c)$ where

$$\begin{bmatrix} 1 & -1 & 1 & -3 \\ -1 & 2 & 1 & 2 \\ 1 & 0 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

   Find the dimension of the kernel (null space) of T and the image (range) of T.

2. Give a direct (ϵ, δ) proof (i.e. just quoting a theorem will not suffice) that the right-hand side of the following displayed formula

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(x/n)}{3^n}$$

   is convergent for all $x \in \mathbb{R}$ and the formula defines a continuous function $f(x)$ on $\mathbb{R}$.

3. a) Show that every group of order 38 has a non-trivial proper normal subgroup N and that N must be abelian.

   b) Find all abelian groups of order 24, up to isomorphism. Explain.

4. Compute the following limit if it exists and justify your conclusion. Note you are not being asked to compute integral, just the limit:

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n(1-x^3)^{1/5}dx$$

5. Let V be the real inner product space of continuous functions on the closed interval $[0, \pi]$ with inner product

$$(f, g) = f \cdot g = \int_0^{\pi} f(x)g(x)dx$$

   Let $W \subset V$ be the subspace of V spanned by the functions 1, $\sin(x)$ and $\cos(x)$.

   a) Find an orthonormal basis of W.

   b) Find the orthogonal projection of the function $f(x) = x$ on W.

6. Consider the set $\mathbb{Q}$ of rational numbers, with the usual topology.

   a) Is every closed and bounded set of $\mathbb{Q}$ compact?

   b) Show that every continuous function $f : \mathbb{R} \to \mathbb{Q}$ is constant.

   Justify your assertions.

Preliminary Examination, Part II

Tuesday afternoon, August 30, 2005

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7. Evaluate the line integral

$$\oint (xe^x + 2xy)dx + (x^2 + x + \cos(y))dy$$

where C is the trapezoid with vertices $(-2, 0)$, $(3, 0)$, $(1, 2)$, $(-1, 2)$, oriented counterclockwise.

8. Suppose f is a twice differentiable function which satisfies the differential equation

$$\frac{d^2 f}{dx^2} = -(2 + e^{-x}) f(x)^2$$

for $x \geq 0$. Suppose $f(0) = 1$ and $f'(0) = 0$. (You are not expected to solve the differential equation.) Sketch the graph of f and show $f(x) = 0$ for one and only one value of x.

9. a) Suppose $p, n \in \mathbb{Z}$ where p is prime and p does not divide n. Must there exist integers a, b such that $ap + bn = 1$?

   b) Suppose that $f, g \in \mathbb{Q}[x]$ where f is irreducible and f does not divide g. Must there exist $h, k \in \mathbb{Q}[x]$ such that $hf + kg = 1$?

   c) Repeat part (b) with $\mathbb{Q}[x]$ replaced by $\mathbb{Q}[x, y]$.

   Justify your assertions.

10. Find the maximum value of $f(x, y, z) = xyz$ for the region

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 3z^2 \leq 18\}.$$

11. a) Find complex numbers a, b, c such that the matrix

$$M = \begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ i & 0 & 0 \end{bmatrix}$$

has the property that $\mathbb{C}^3$ has an orthonormal basis B consisting of eigenvectors of M. (Here, we use the standard inner product of $\mathbb{C}^3$).

b) For your choice of a, b, c find such an orthonormal basis B, and find the corresponding eigenvalues of M.

12. Suppose f is a uniformly continuous function on $[0, \infty)$. Prove

$$\lim_{x \to \infty} \frac{f(x)}{1 + x^2} = 0 \quad .$$