This part of the examination consists of six problems. You should work on all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.
1. Let $M$ be a compact metric space and suppose $T : M \to M$ is a continuous function such that
   \[ d(T(x), T(y)) < d(x, y) \]
   for every $x, y \in M$ such that $x \neq y$.
   (a) Prove that $T$ has a unique fixed point. (Hint: Minimize $d(T(x), x)$.)
   (b) Prove that if $x_0 \in M$ is any point, then the sequence $\{x_n\}_{n=1}^{\infty}$, defined inductively by $x_1 = T(x_0)$, $x_{n+1} = T(x_n)$ converges to the fixed point.

2. Suppose $n \geq 2$ is an integer. Which of the following subsets of $G = GL(n, \mathbb{R})$ is a subgroup? Which subsets are normal subgroups? Explain.
   (a) $\{A \in G \mid \text{all entries of } A \text{ are integers}\}$
   (b) $\{A \in G \mid \det(A) > 0\}$
   (c) $\{A \in G \mid A = A^t\}$
   (d) $\{A \in G \mid A \cdot A^t = I\}$

3. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a continuous function.
   (a) Show that $f$ can have at most countably many strict local maxima.
   (b) Assume that $f$ is not monotone on any interval. Then show that the local maxima of $f$ are dense in $\mathbb{R}$.

4. Show that the ring $\mathbb{Z}[[x]]$ (of formal power series $\sum_{i=0}^{\infty} a_i x^i$ with integers $a_i$) is an integral domain. Which elements in this ring are invertible (under multiplication)?

5. For any real number $\alpha \in \mathbb{R}$ define the sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = \alpha$, $x_{n+1} = x_n^2 - 1$.
   (a) Prove that if the above sequence is convergent, then the limit has to be either $(1 + \sqrt{5})/2$ or $(1 - \sqrt{5})/2$.
   (b) Determine whether the sequence converges for all real numbers $\alpha$, for no real numbers $\alpha$, or for some but not all real numbers $\alpha$. 

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6. Let $f : [0,1] \to [0,\infty)$ be a smooth function such that $f(0) = f(1) = 0$ and let $C$ be the smooth curve from $(0,0)$ to $(1,0)$ given by the graph of $f$. Suppose that the line integral

$$\int_C (y + x)dx + ydy = \frac{3}{4}.$$ 

Find

$$\int_C y(e^{xy} + 1)dx + xe^{xy}dy.$$ 

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7. Let $D$ be a compact subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. The graph of $f$ is the set

$$G(f) = \{(x, f(x)) : x \in D\} \subset \mathbb{R}^3$$

Show that $f$ is continuous if and only if $G(f)$ is compact.

8. For any real number $c$, let $V_c$ be the set of solutions to the differential equation $y'' - 4y' + 4y = c$.

(a) Find all $c$ such that $V_c$ is a vector space. For each such $c$, find an explicit basis of $V_c$.

(b) Show that if $V_c$ is a vector space, then differentiation is a linear transformation $D : V_c \to V_c$. Find the matrix of $D$ with respect to your basis in (a).

9. Suppose $x_1, x_2, x_3 \in \mathbb{R}^2$ are three points in the plane that do not lie on a line, and denote by $T$ the closed triangle with vertices $x_1$, $x_2$ and $x_3$. Suppose $f : T \to \mathbb{R}$ is a continuous function which is differentiable on the interior of $T$ and which vanishes on the boundary of $T$.

Prove that there exists a point $x$ in the interior of $T$ such that the tangent plane to the graph of $f$ at the point $x$ is horizontal.

10. Let $R = (\mathbb{Z}/2)[x]$ and let $I$ be the ideal $I = R \cdot (x^{17} - 1)$. Is there a non-zero element $a$ of the quotient ring $A = R/I$ such that $a^2 = 0$?

11. Let $f : S^1 \to S^1$ be a continuous function from the unit circle to itself. Prove that if $f$ is not onto, then $f$ must have a fixed point.

12. Let $T$ be the transpose map $T(A) = A^{tr}$ for $A \in M_2(\mathbb{R})$. Find the eigenvalues of $T$ and a basis of $M_2(\mathbb{R})$ with respect to which $T$ is diagonal.