Preliminary Examination, Part I

August 30, 2011

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

Part I

1. Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of non-negative continuous real-valued functions on $\mathbb{R}$ such that $f_{n+1}(x) \leq f_n(x)$ for all $n \geq 1$ and all $x \in \mathbb{R}$.

a) Prove that there is a unique function $f(x)$ on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.

b) Must $f_n \to f$ uniformly on the closed interval $[0, 1]$?

2. For $x, y \in \mathbb{R}$, let $d(x, y) = |x - y|/(1 + |x - y|)$. Determine whether d defines a metric on $\mathbb{R}$. If it does, determine whether $\mathbb{R}$ is compact under d.

3. Let $\|x\|$ denote the Euclidean norm of a (column) vector in $\mathbb{R}^n$. For an $n \times n$ real matrix $A = (a_{ij})$, define

$$\|A\| = \sup\{\|Ax\|/\|x\| \mid x \in \mathbb{R}^n, x \neq 0\}.$$

a) Prove that $\|A\|$ exists and is a non-negative real number for all A, with $\|A\| = 0$ if and only if A is the zero matrix.

b) Prove that $\|A + B\| \leq \|A\| + \|B\|$ and that $\|AB\| \leq \|A\| \, \|B\|$ for any two $n \times n$ real matrices A, B.

c) Let A be an $n \times n$ real matrix such that $\|A\| < 1$. Prove that

$$\frac{1}{1 + \|A\|} \leq \|(I - A)^{-1}\| \leq \frac{1}{1 - \|A\|}.$$

(Hint: First show that $(I - A)^{-1} = I + A(I - A)^{-1}$.)

4. Let C be the solid cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$. Evaluate

$$\iiint_C (x^2 + y^2 + z^2)z \, dV.$$

5. Let G be a non-abelian simple group. Let H be a subgroup of index 7 in G. Determine the number of conjugates of H in G. Justify your assertion.

6. Using just the ε-δ definition, show that the function $f(x) = x^2$ is continuous on $\mathbb{R}$.

Preliminary Examination, Part II
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Part II

7. Evaluate

$$\oint_C (\sin x \, \sin y)dx + (x - \cos x \, \cos y)dy,$$

where C is the circle of radius 2 about the origin, oriented counterclockwise.

8. Let f be a real-valued function defined on $\mathbb{R}$. Suppose that $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\big(\frac{i}{n}\big)$ exists and is equal to 2.

a) Must $\int_0^1 f(x)\, dx$ exist? If it exists, must it be equal to 2?

b) Redo part (a) under the additional assumption that f is continuous. (You may use standard theorems in answering this part.)

9. Let m be a positive integer. Let R_m be the subset of $\mathbb{Q}$ consisting of the rational numbers that can be written in the form a/b, where a, b are integers and b is not divisible by m.

a) Determine if R_5 is a commutative ring. If it is, determine if it is an integral domain and find all of its maximal ideals.

b) Do the same for R_6.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is strictly increasing (i.e. $f(x) < f(y)$ whenever $x < y$). Find a necessary and sufficient condition for f to have a strictly increasing continuous inverse function $g : \mathbb{R} \to \mathbb{R}$. Prove your assertion.

11. Let X be a metric space, and let Y be a subset of X with the induced metric. Just using the definitions, prove that if Y is compact then Y is a closed subset of X.

12. Let V be the vector space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $L : V \to V$ be the linear operator that sends f to $f'' + f'$. Find all real eigenvalues of L and the corresponding eigenvectors.