Preliminary Examination, Part I

Wednesday, August 26, 2015

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name on each workbook you submit. All problems have equal weight.

Prelim Exam: Fall 2015.
Part I: Wednesday

1. Determine whether any of the following three matrices are similar over $\mathbb{R}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

2. Let $f := \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$|f'(x)| \leq 2015 \qquad \text{for all } x \in \mathbb{R}.$$

Prove that f is uniformly continuous on $\mathbb{R}$.

3. Let $F = \mathbb{Z}/3$, and let $R = F[x]/(x^3 - x - 1)$.

(a) Find the number of elements of the ring R.

(b) Is R an integral domain? Is it a field? Which primitive roots of unity does it contain?

4. Let $f : (0, \infty) \to (0, \infty)$, be a continuous and decreasing function. Prove that the sequence

$$f(1) + \cdots + f(n) - \int_1^{n+1} f(t)\, dt$$

is convergent and that its limit lies in the closed interval $[0, f(1)]$.

5. Let V be the space of $n \times n$ matrices over a field K. Let A be an $n \times n$ matrix. Let $T_A : V \to V$ be the linear transformation given by $T_A(B) = AB$.

(a) Show that if $F(x)$ is a polynomial over K, then $F(A) = 0$ if and only if $F(T_A) = 0$.

(b) Show that T_A and A have the same minimal polynomial. Is this also true for the characteristic polynomials of T_A and A?

6. Consider the vector field $\vec{F} = (\pi y e^{\sin x} \cos x - z)\vec{i} + (\pi e^{\sin x} - x)\vec{j}$, and the curve C given by

$$\vec{r} = (2\cos t)\vec{i} + (2\sin t)\vec{j} + t\vec{k},$$

for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

Preliminary Examination, Part II

Thursday, August 27, 2015

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name on each workbook you submit. All problems have equal weight.

Prelim Exam: Fall 2015
Part II: Thursday

7. Let X be a topological space, and let Y, Z be connected subsets of X such that $X = Y \cup Z$ and such that $Y \cap Z$ is not empty. Prove that X is connected.

8. Let $\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be the vector valued function that solves the initial value problem
$$\mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Calculate $\mathbf{x}$.

9. Produce a 3×3 real matrix A that has all the following three properties:

   (a) The set of eigenvalues is $\{-1, 2\}$;

   (b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector for the eigenvalue -1;

   (c) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors for the eigenvalue 2.

10. For each of the following, determine whether the sequence $a_1, a_2, a_3, \ldots$ converges. If the sequence converges, determine the limit. Justify your assertions.

    (a) $a_n = n^{100}/10^n$ for $n = 1, 2, 3, ...$

    (b) $a_1 = 0$, $a_{n+1} = (2a_n + 1)/3$ for $n \geq 1$.

11. Let G be the dihedral group of order 50, generated by elements a, b subject to the relations $a^{25} = 1$, $b^2 = 1$, and $ba = a^{-1}b$.

    (a) For which integers n does G have an element of order n?

    (b) Find all normal subgroups of G.

12. Say $f(x) \geq 0$ is a continuous function and $\int_0^1 f(x)dx = 0$. Either prove that $f(x) = 0$ for all $0 \leq x \leq 1$, or provide a counterexample.