FALL 2015 PRELIMINARY EXAM SOLUTIONS

Problem 1. Recall that two matrices A, B are similar if there exists an invertible matrix S such that $A = SBS^{-1}$. Similar matrices have the same eigenvalues, and these have the same geometric multiplicity (dimension of eigenspace). No two of the three given matrices are similar, because:

- $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has unique eigenvalue 2, whose eigenspace is 2 dimensional;
- $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ has unique eigenvalue 2, whose eigenspace is 1 dimensional, spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
- $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ has eigenvalues ± 2.

Problem 2. Recall the definition of uniform continuity, especially the order of qualifiers: for all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x_0 \in \mathbb{R}$:

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon. \tag{0.1}$$

So assume given $\epsilon > 0$ and $x_0 \in \mathbb{R}$; we need to find $\delta > 0$, which depends on ϵ, but *not* on x_0, and makes 0.1 true. [1] Using the mean value theorem, for any $x \in \mathbb{R}$, there exists $c \in \mathbb{R}$ such that:

$$|f(x) - f(x_0)| = |f'(c)||x - x_0| \leq 2015|x - x_0|.$$

Taking $\delta = \epsilon/2015$ gives 0.1.

Problem 3. R is obtained from $F[x]$ by imposing the relation $x^3 = x + 1$. Using this, we can express all powers of x that are ≥ 3 as linear combinations of $1, x, x^2$. As such, R is a 3 dimensional vector space over $\mathbb{Z}/3$, and it has $3^3 = 27$ elements.

We check explicitly that the polynomial $x^3 - x - 1$ has no roots in $\mathbb{Z}/3$. Since it has degree 3, the ideal $(x^3 - x - 1)$ is prime, and R is an integral domain. In fact, $F[x]$ is a PID, so every prime ideal is maximal. It follows that R is a field.

Since R is a field with 27 elements, the multiplicative group $R^\times$ has 26 elements (remove 0). $R^\times$ is abelian, so there exists a group isomorphism $\phi : (\mathbb{Z}/26, +) \to R^\times$. [2] We conclude:

- $\phi(13) = 2$ is a primitive square root of unity;
- $\phi(k)$ is a primitive 13^{th} root of unity for even k;
- $\phi(k)$ is a primitive 26^{th} root of unity for k odd, other than 13.

Problem 4. First, write the sequence as:

$$S(n) = \sum_{i=1}^{n} \left[f(i) - \int_i^{i+1} f(t)dt \right].$$

Note that f is decreasing, so $\int_i^{i+1} f(t)dt \leq \int_i^{i+1} f(i)dt = f(i)$. Thus, each term in the sum is non-negative. Consequently:

- $S(n) \geq 0$;
- $S(n)$ is increasing.

Next, write the sequence as:

$$S(n) = f(1) + \sum_{i=1}^{n-1} \left[f(i+1) - \int_i^{i+1} f(t)dt \right] - \int_n^{n+1} f(t)dt.$$

Since f is decreasing, $\int_i^{i+1} f(t)dt \geq \int_i^{i+1} f(i+1)dt = f(i+1)$. Moreover, $f(t) > 0$ for all t, so $\int_n^{n+1} f(t)dt > 0$. Consequently:

[1] If we allowed δ to depend on x_0 as well, we would obtain continuity, but not uniform continuity.

[2] Each choice of generator of $R^\times$ gives a different ϕ; for example one can take $\phi(1) = 2x$.

- $S(n) \leq f(1)$.

Any series that is monotonous and bounded is convergent. So the three bullet points imply that S is convergent, with limit contained in $[0, f(1)]$.

Problem 5.

(a) The main point is that matrix multiplication coincides with the composition of the associated linear operators: $F(T_A) = T_{F(A)}$. So if $F(A) = 0$, it is immediate that $F(T_A) = 0$. Conversely, assume that $F(A)B = 0$ for every $n \times n$ matrix B. In particular, fix $1 \leq i \leq n$, and let B be the matrix with $B_{i1} = 1$, and all other entries 0. Then $F(A)B = 0$ implies that the i^{th} column of $F(A)$ is 0. Repeating for all i, we obtain $F(A) = 0$.

(b) The minimal polynomial of A is the polynomial F of minimal degree such that $F(A) = 0$. From part a, it's immediate that the minimal polynomials of A and T_A coincide. However, A is a linear operator on an n dimensional vector space, and T_A is a linear operator on an n^2 dimensional vector space. Their characteristic polynomials have degrees n, n^2 respectively. They cannot be equal unless $n = 1$.

Problem 6. Write $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, where:

$$\begin{aligned} \mathbf{F}_1 &= \pi y e^{\sin x} \cos x \mathbf{i} + \pi e^{\sin x} \mathbf{j} \\ \mathbf{F}_2 &= -z\mathbf{i} - x\mathbf{j} \end{aligned}$$

Notice that $\mathbf{F}_1 = \nabla(\pi y e^{\sin x})$, and $\mathbf{F}_1$ doesn't depend on z. As such, its integral along C is equal to the integral along the projection of C in the xy plane, which is a closed curve. The fundamental theorem of calculus then implies $\int_C \mathbf{F}_1 \cdot d\mathbf{r} = 0$. Then:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F}_2 \cdot d\mathbf{r} \\ &= \int_C (-z\mathbf{i} - x\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_C -zdx - xdy \\ &= \int_{-\pi/2}^{\pi/2} 2t \sin t dt - 4\cos^2 t dt \\ &= 4 - 2\pi. \end{aligned}$$

Problem 7. Let $U, V \subset X$ be disjoint open subsets such that $U \cup V = X$. We need to prove that either $U = X$ or $V = X$. Start from:

$$(U \cap Y) \cup (V \cap Y) = Y.$$

$(U \cap Y)$ and $(V \cap Y)$ are disjoint open subsets of Y, whose union is Y. Y is connected by hypothesis, so either $U \cap Y = Y$ or $V \cap Y = Y$. Without loss of generality $U \cap Y = Y$, i.e. $U \supset Y$.

Analogously,

$$(U \cap Z) \cup (V \cap Z) = Z$$

implies that either $U \cap Z = Z$ or $V \cap Z = Z$.

Assume that $V \cap Z = Z$, which means that $V \supset Z$. Then $\emptyset = U \cap V \supset Y \cap Z \neq \emptyset$, a contradiction. So the only possibility is $U \cap Z = Z$. But then U contains both Y and Z, so $U = X$.

Problem 8. The general solution to the given ODE is:

$$\mathbf{x}(t) = \exp t \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix} \mathbf{x}(0).$$

Our goal is to compute the exponential. We decompose the given matrix into two commuting terms:

$$\begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} =: D + N.$$

Due to commutativity:

$$\exp t \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix} = e^{tD} e^{tN}.$$

These factors are easy to compute. D is a multiple of the identity matrix I, so $e^{tD} = e^{-t}I$. $N^2 = 0$, so $e^{tN} = I + tN$. Then:

$$\mathbf{x}(t) = e^{-t}\begin{bmatrix} 1 & 0 \\ 4t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 4te^{-t} \end{bmatrix}.$$

Problem 9. Let S denote the matrix whose columns are the given eigenvectors:

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

Then the following matrix has the required properties:

$$A = S\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 3 & -3 \\ -6 & 10 & -6 \\ -9 & 9 & -5 \end{bmatrix}.$$

This happens because S is the change of basis matrix between the standard basis of $\mathbb{R}^3$ and the basis of eigenvectors.

Problem 10.

(a) The ratio of two consecutive terms is:

$$\frac{a_{n+1}}{a_n} = \frac{(1 + 1/n)^{100}}{10}.$$

For $1 + 1/n < 5^{1/100}$, which covers all but finitely many n, this is smaller than $1/2$. Comparison with $(1/2)^n$ shows that the sequence converges to 0.

(b) We'll show that the sequence is bounded and increasing, hence convergent. First, we prove by induction that $0 \leq a_n < 1$. For a_1 this is clear. Assuming that the statement holds for a_n, we have:

$$\frac{1}{3} \leq a_{n+1} < \frac{3}{3},$$

so the statement holds for a_{n+1} as well.

Consider now:

$$a_{n+1} - a_n = \frac{1 - a_n}{3}.$$

We know that $a_n < 1$, so $a_{n+1} - a_n > 0$, i.e. the sequence is increasing.

We proved that the limit exists; now we can compute it by solving the recurrence relation:

$$\ell = \frac{2\ell + 1}{3} \Longrightarrow \ell = 1.$$

Problem 11.

(a) Throughout we use the fact that $b = b^{-1}$. All elements of G are of the form a^i or ba^i, for some $0 \leq i \leq 24$. By explicit computation using the group relations we see that:

- a^0 has order 1;
- a^i has order 5 if i is a nonzero multiple of 5;
- a^i has order 25 if i is not a multiple of 5;
- ba^i has order 2 for all i.

We have exhausted all elements, so the possible orders are $1, 2, 5, 25$.

(b) We use the following conjugation relations:

$$ba^ib^{-1} = a^{-i} \tag{0.2}$$

$$b(ba^i)b^{-1} = ba^{-i} \tag{0.3}$$

$$aba^{-1} = ba^{-2} \tag{0.4}$$

From 0.2 we see that the subgroups $\langle a\rangle$ and $\langle a^5\rangle$ are normal. From 0.3 we see that the order 2 subgroups $\langle ba^i\rangle$ are not normal for any $i \neq 0$. From 0.4 we see that $\langle b\rangle$ is not normal either.

We can also consider subgroups generated by multiple elements. $\langle b, a^5\rangle$ is a subgroup of order 10, but it's not normal due to 0.4. Any other choice of generators produces one of the subgroups already considered.

So the normal subgroups are $1, \langle a \rangle, \langle a^5 \rangle, G$.

Problem 12. Since $f \geq 0$, we have that $\int_0^t f(x)dx \geq 0$ and $\int_t^1 f(x)dx \geq 0$, for all $0 \leq t \leq 1$. This gives two non-negative quantities which sum to 0:

$$\int_0^t f(x)dx + \int_t^1 f(x)dx = \int_0^1 f(x)dx = 0.$$

So $\int_0^t f(x)dx = 0$. But for all $0 \leq t < 1$:

$$f(t) = \lim_{\epsilon \to 0} \frac{\int_0^{t+\epsilon} f(x)dx - \int_0^t f(x)dx}{\epsilon} = \lim_{\epsilon \to 0} 0 = 0.$$

For $t = 1$, the above argument doesn't work verbatim, because we don't know the behavior of f at $1 + \epsilon$. We can replace it with:

$$f(1) = \lim_{\epsilon \to 0} \frac{\int_0^1 f(x)dx - \int_0^{1-\epsilon} f(x)dx}{\epsilon} = \lim_{\epsilon \to 0} 0 = 0.$$