Signature

PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Monday, August 29, 2016 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight.

Score	
1	
2	
3	
4	
5	
6	
Total 1–6	
Total 7–12	
Total	

1. Let V be the real vector space of continuous real-valued functions on the closed interval $[0,1]$, and let $w \in V$. For $p, q \in V$, define $\langle p, q \rangle = \displaystyle\int_0^1 p(x)q(x)w(x)\,dx$.

   a) Suppose that $w(a) > 0$ for all $a \in [0,1]$. Does it follow that the above defines an inner product on V? Justify your assertion.

   b) Does there exist a choice of w such that $w(1/2) < 0$ and such that the above defines an inner product on V? Justify your assertion.

2. Let $\{x_n\}$ be a sequence of real numbers (indexed by $n \geq 0$), and let $0 < c < 1$ be a real number. Suppose that

$$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|$$

for all $n = 1, 2, 3, \ldots$.

a) If $n \geq k$ are positive integers, show that

$$|x_{n+1} - x_k| < \frac{c^k}{1-c}\,|x_1 - x_0|.$$

(Hint: First bound $|x_{n+1} - x_n|$ in terms of $|x_1 - x_0|$.)

b) Prove that the sequence $\{x_n\}$ converges to a real number.

3. a) In the polynomial ring $\mathbb{Q}[x]$, consider the ideal I generated by $x^4 - 1$ and $x^3 - x$. Does I have a generator $f(x) \in \mathbb{Q}[x]$? Either find one or explain why none exists.

b) In the polynomial ring $\mathbb{Q}[x, y]$, do the same for the ideal generated by the polynomials x and y.

4. For each of the following, give either a proof or a counterexample.

   a) Let f be a continuous real-valued function on the open interval $0 < x < 3$. Must f be uniformly continuous on the open interval $1 < x < 2$?

   b) Suppose instead that f is only assumed to be continuous on the open interval $0 < x < 2$. Must f be uniformly continuous on the open interval $1 < x < 2$?

5. Let V, W be two-dimensional real vector spaces, and let $f_1, \ldots, f_5$ be linear transformations from V to W. Show that there exist real numbers $a_1, \ldots, a_5$, not all zero, such that $a_1 f_1 + \cdots + a_5 f_5$ is the zero transformation.

6. Evaluate $\oint_C \left(e^{x^2} + \sin(y^2)\right) dx + \left(2xy\cos(y^2) + xy^3\right) dy$, where C is the triangle with vertices $(0,0), (1,-1), (1,1)$, oriented counterclockwise.

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Preliminary Examination, Part II

Monday, August 29, 2016 | 1:30-4:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight.

Score	
7	
8	
9	
10	
11	
12	
Total 7–12	

7. Let $f : X \to Y$ be a continuous map between metric spaces. For each of the following, give either a proof or a counterexample, using just the definition of compactness.

   a) If $A \subseteq X$ is compact, then so is $f(A) \subseteq Y$.

   b) If $B \subseteq Y$ is compact, then so is $f^{-1}(B) \subseteq X$.

8. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a constant A such that

$$\int_0^x f(t)(1+t^2)dt = \cos(x^2) + A.$$

9. For every integer $n > 1$, let U_n be the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$ under multiplication.

   a) Find the orders of U_8 and U_9. Explain.

   b) Determine whether the groups U_8 and U_9 are cyclic.

10. Let $f(x, y) = x^2 - xy + y^2 - y$.

a) Does the function f achieve an absolute maximum on $\mathbb{R}^2$? an absolute minimum on $\mathbb{R}^2$? If so, find all points where this occurs.

b) Do the same with $\mathbb{R}^2$ replaced by the square $0 \le x, y \le 1$.

11. Let a, b, c be real numbers, and consider the matrix $A = \begin{pmatrix} a & b & c \\ b & c & b \\ c & b & a \end{pmatrix}$.

a) Explain why all the eigenvalues of A must be real.

b) Show that some eigenvalue λ of A has the property that for every vector $v \in \mathbb{R}^3$, $\quad v \cdot Av \leq \lambda \|v\|^2$. (Note: You are not being asked to compute the eigenvalues of A.)

12. Consider the differential equation $y^{(4)} - y = ce^{2x}$ where c is a real constant.

a) Let S_c be the set of solutions of this equation. For which c is this set a vector space? Why?.

b) For each such c, find this solution space explicitly, and find a basis for it.