Fall 2017 Prelim answers

Part I:

1. Find an orthogonal basis of $\mathbb{R}^3$ that contains a basis of the span of $(1,2,3)$ and $(4,5,6)$.

Solution:

First use Gram-Schmidt to find an orthogonal basis $\{v_1, v_2\}$ for the span of $w_1 = (1,2,3)$ and $w_2 = (4,5,6)$. We may take $v_1 = w_1$. The orthogonal projection of w_2 on v_1 is $pr_{v_1} w_2 = [(v_1 \cdot w_2)/(v_1 \cdot v_1)]v_1 = (16/7)v_1$. So the vector $v_2' := w_2 - pr_{v_1} w_2 = (12/7, 3/7, -6/7)$ is orthogonal to v_1 and is in the span of w_1, w_2. We may take $v_2 = (7/3)v_2' = (4, 1, -2)$. Now to find a third orthogonal basis vector for $\mathbb{R}^3$ that includes v_1, v_2, we can take their cross product $v_1 \times v_2 = (-7, 14, -7)$, or any non-zero multiple of this. So we can take $v_3 = (1, -2, 1)$.

2. For each positive integer n, define $f_n(x) = x^n$ for $0 \le x \le 1$.

(a) Is each function f_n uniformly continuous?

(b) Is the sequence of functions $\{f_n\}$ uniformly convergent?

Justify your assertions.

Solution:

(a) Yes. The function f_n is continuous because it is a polynomial, and it is uniformly continuous because it is given on a closed interval.

(b) No. The functions f_n converge pointwise to the function given by $f(x) = 0$ for $0 \le x \le 1$ and $f(1) = 1$. The function f is discontinuous. Since the uniform limit of continuous functions is continuous, the functions f_n do not converge uniformly.

3. (a) How many abelian groups of order 108 are there, up to isomorphism?

(b) Are there any non-abelian groups of order 108? Either show that there aren't any or else give an example of one.

Solution:

(a) $108 = 2^2 \times 3^3$. By the fundamental theorem of finite abelian groups, a group of order 108 is of the form $A \times B$, where A is abelian of order 2^2 and B is abelian of order 3^3. Here A is a direct product of cyclic 2-groups and B is a direct product of cyclic 3-groups. Since there are 2 partitions of 2 and 3 partitions of 3, there are $6 = 2 \cdot 4$ abelian groups of order 108: $\mathbb{Z}/4 \times \mathbb{Z}/27; \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/27; \mathbb{Z}/4 \times \mathbb{Z}/9 \times \mathbb{Z}/3; \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/9 \times \mathbb{Z}/3; \mathbb{Z}/4 \times (\mathbb{Z}/3)^3; \mathbb{Z}/2 \times \mathbb{Z}/2 \times (\mathbb{Z}/3)^3$.

(b) There is a dihedral group of this order, on generators a, b with relations $a^{54} = 1, b^2 = 1, bab^{-1} = a^{-1}$. Another possible example is the product of the symmetric group S_3 with the cyclic group of order 18. (There are various others.)

4. Let $\{a_1, a_2, \ldots\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge. Let $b_1, b_2, \ldots$ be the positive terms among the a_n's, and let $c_1, c_2, \ldots$ be the negative terms.

(a) Prove that there are infinitely many terms b_i and infinitely many terms c_i.

(b) Prove that the series $\sum_{i=1}^{\infty} b_i$ diverges to ∞, and $\sum_{i=1}^{\infty} c_i$ diverges to $-\infty$.

(c) Let α be a real number. Show that there is some rearrangement of the terms a_n such that the sum of the rearranged series converges to α.

Solution:

(a) If there are only finitely many terms c_i, then after omitting a finite number of initial terms of the sequence $\{a_n\}$, we may assume that all a_n are positive and so $a_n = |a_n|$. This contradicts the assumption that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge. The case of only finitely many b_i is similar.

(b) If $\sum_{i=1}^{\infty} c_i$ converges, say to $c < 0$, then in each partial sum of $\sum_{n=1}^{\infty} a_n$ the sum of the negative terms is at least c. Thus for each N, $\sum_{n=1}^{N} a_n \geq \sum_{n=1}^{M} b_n + c$, where M is the number of non-negative terms among $a_1, \ldots, a_N$. Since $\sum_{n=1}^{\infty} |a_n|$ does not converge and the terms are positive, the partial sums become arbitrarily large. Hence so do the partial sums of $\sum_{i=1}^{\infty} b_i$. Thus that series diverges to ∞. The case of $\sum_{i=1}^{\infty} c_i$ is similar.

(c) Begin by choosing terms $b_1, b_2, \ldots, b_{n_1}$, until the partial sum first reaches a number $\beta_1 \geq \alpha$; we can do this since the sum of the b_i diverges to ∞. Here $\beta_1 < \alpha + b_{n_1}$. Next choose terms $c_1, c_2, \ldots, c_{n_2}$, following the terms we have so far (which add to β_1) until we first reach a number $\beta_2 \leq \alpha$; we can do this since the sum of the c_i diverges to $-\infty$. Here $\beta_2 > \alpha + c_{n_2}$. Then choose the next terms in the $b_{n_1+1}, \ldots, b_{n_3}$ to get a sum $\beta_3 \geq \alpha$, etc. For each odd k, $\alpha \leq \beta_k < \alpha + b_{n_k}$; and for each even k, $\alpha \geq \beta_k > \alpha + c_{n_k}$. The partial sums appearing after β_k and before β_{k+1} lie between β_k and β_{k+1}. Since the series $\sum_{n=1}^{\infty} a_n$ converges, the terms $a_n \to 0$, and hence $b_i \to 0$ and $c_i \to 0$. Thus the partial sums of the rearranged series converge to α.

5. Let A denote the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

(a) Determine if there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A. If there is, find one.

(b) Compute $A^{2017}u_0$, where $u_0 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

Hint: Do not try to compute this directly.

Solution:

(a) The characteristic polynomial of A is $(4-\lambda)(2-\lambda) - 3 \cdot 1 = (\lambda - 5)(\lambda - 1)$, so the eigenvalues are $1, 5$. Since these are distinct, there is a basis of eigenvectors. Explicitly, $(1, -1)$ is an eigenvector with eigenvalue 1, and $(3, 1)$ is an eigenvector with eigenvalue 5, since these span the kernels of $A - I$ and $A_5 I$ respectively. One sees directly that these two vectors are linearly independent, and so form a basis of $\mathbb{R}^2$.

(b) By the explicit choice of eigenvectors in (a), one has

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}^{-1}.$$

So

$$A^{2017} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{2017} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + 3 \cdot 5^{2017} \\ -1 + 5^{2017} \end{bmatrix}.$$

6. (a) Show that a closed subset of a compact topological space is compact.

(b) Show that a compact subset of a Hausdorff space is closed.

Solution:

(a) Let $\{U_i\}_{i \in I}$ be an open cover of a closed subset A of a topological space X, with $U_i \subseteq A$. Since A is given the subspace topology, for each i there is an open set $\tilde{U}_i \subset X$ such that $U_i = \tilde{U}_i \cap A$. The sets $\tilde{U}_i$, together with the complement $\tilde{U}$ of A in X, form an open cover of X. Since X is compact, there is a finite subcover consisting of $\tilde{U}_j$ for all $j \in J$ (where J is a finite subset of I), possibly together with $\tilde{U}$. Since $\tilde{U}$ is disjoint from A, the sets $U_j = \tilde{U}_j \cap A$, for $j \in J$, form a finite subcover of A.

(b) Let $A \subset Y$, with A compact and Y Hausdorff. Suppose $y \notin A$. For each $x \in A$ there exist open sets U_x and V_x with $x \in U_x$, $y \in V_x$, and $U_x \cap V_x = \emptyset$. The sets U_x together form an open cover of A, and so there exist $x_1, \ldots, x_n$ such that $A \subset U_{x_1} \cup ... \cup U_{x_n}$. Let $V = V_{x_1} \cap ... \cap V_{x_n}$. Then $y \in V$, V is open, and $V \cap A = \emptyset$. This shows that every point of $X \setminus A$ is contained in an open set that is disjoint from A. Hence $X \setminus A$ is open, and so A is closed.

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Part II:

7. Evaluate the contour integral

$$\oint_C \big(y^3 + 3x^2y + \cos(x^2)\big)dx + \big(x + e^{y^3}\big)dy,$$

where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise. (Hint: Some ways are easier than others.)

Solution:

$C = \partial D$, where D is the closed unit disc $x^2 + y^2 \leq 1$. By Green's theorem,

$$\begin{aligned}\oint_C \big(y^3 + 3x^2y + \cos(x^2)\big)dx + \big(x + e^{y^3}\big)dy &= \iint_D \frac{\partial}{\partial x}(x + e^{y^3}\big) - \frac{\partial}{\partial y}(y^3 + 3x^2y + \cos(x^2)\big)dx\,dy \\ &= \iint_D 1 - (3y^2 + 3x^2)dx\,dy \\ &= \iint_D 1\,dx\,dy - \iint_D 3r^2\ r\ dr\ d\theta \\ &= \pi - 2\pi \cdot 3/4 = -\pi/2.\end{aligned}$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that $0 < f(x) < 1$ for all real numbers x. Show that $f''(x) = 0$ for some real number x.

Solution:

Since f is infinitely differentiable, f'' is continuous. Suppose that f'' is never equal to 0. By the intermediate value theorem, f'' is either always positive or always negative. Possibly after replacing $f(x)$ by $1-f(x)$, we may assume that f'' is always positive. So f' is a strictly increasing function, and is therefore non-constant. Therefore $f'(a)$ is non-zero for some a. After replacing $f(x)$ by $f(-x)$ (which does not affect the condition that f'' is always positive), we may assume that $f'(a) > 0$ for some a. Let $b = a + 1/f'(a)$. So $b > a$. By the mean value theorem, there exists c with $a \leq c \leq b$ such that $f'(c) = (f(b)-f(a))/(b-a) = f'(a)(f(b) - f(a))$. Since f' is increasing, $f'(a) \leq f'(c) = f'(a)(f(b) - f(a))$, and thus $1 \leq f(b) - f(a) < f(b)$ since $f(a) > 0$, and this is a contradiction.

9. Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear transformation, corresponding to a matrix A. Let T^* be the adjoint operator of T, corresponding to the transpose of A. Show that

$$\ker(T^*T) = \ker(T).$$

Here $\ker(T)$ denotes the kernel of T.
(Hint: Consider $\|Tx\|$.)

Solution: If $x \in \ker(T)$, then $T^*T(x) = T^*(0) = 0$ and so $x \in \ker(T^*T)$. To prove the opposite inclusion, let $x \in \ker(T^*T)$. Then

$$\langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = 0;$$

i.e., $\|Tx\| = 0$, and so $Tx = 0$ and $x \in \ker(T)$.

10. Using just the definition of the derivative, prove that every differentiable function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution:

For every real number a, there is a derivative of f at a, and

$$\lim_{h\to 0}(f(a+h) - f(a))/h = f'(a).$$

So $\lim_{h\to 0} f(a+h) - f(a) = \lim_{h\to 0} hf'(a) = 0$. Thus $\lim_{b\to a} f(b) = \lim_{h\to 0} f(a+h) = f(a)$.

11. (a) For which integers n is there a finite field whose additive group is cyclic of order n?

(b) For which integers n is there a finite field whose multiplicative group of invertible elements is cyclic of order n?

Justify your assertions.

Solution:

(a) These are precisely the prime numbers. A finite field has prime characteristic p, and is a vector space over the field of p elements, say of dimension r. The additive group is thus isomorphic to $(\mathbb{Z}/p)^r$, of order $n = p^r$. This is cyclic iff $r = 1$, i.e. iff n is the prime p.

(b) These are the integers of the form $p^r - 1$, for p prime and r a positive integer. A finite field F has order p^r for some prime p and $r \geq 1$, and for every p, r there is such a field. The multiplicative group $F \smallsetminus \{0\}$ of a finite field is cyclic, of order one less than the order of F. So the assertion follows.

12. Find orthogonal trajectories for the family of plane curves E_c given by $4x^2+9y^2 = c$, for $c > 0$. That is, find a non-constant one-parameter family of curves D_t such that each D_t intersects each E_c orthogonally, wherever they meet.

Solution:

The curve E_c satisfies $8x\,dx + 18y\,dy = 0$, i.e., $dy/dx = -4x/9y$ for $y \neq 0$. An orthogonal trajectory has derivative equal to the negative reciprocal, i.e., $dy/dx = 9y/4x$ for $x, y \neq 0$. Solving this differential equation by separation of variables gives $4 \log|y| = 9 \log|x| + C$, or equivalently the curves D_t given by $y^4 = tx^9$ for any non-zero t. (The coordinate axes are also orthogonal to each E_c.)