Preliminary Examination, Part I

Monday, August 29, 2016 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight.

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1. Let $V$ be the real vector space of continuous real-valued functions on the closed interval $[0, 1]$, and let $w \in V$. For $p, q \in V$, define $\langle p, q \rangle = \int_{0}^{1} p(x)q(x)w(x) \, dx$.

a) Suppose that $w(a) > 0$ for all $a \in [0, 1]$. Does it follow that the above defines an inner product on $V$? Justify your assertion.

b) Does there exist a choice of $w$ such that $w(1/2) < 0$ and such that the above defines an inner product on $V$? Justify your assertion.
2. Let \( \{x_n\} \) be a sequence of real numbers (indexed by \( n \geq 0 \)), and let \( 0 < c < 1 \) be a real number. Suppose that

\[
|x_{n+1} - x_n| \leq c|x_n - x_{n-1}|
\]

for all \( n = 1, 2, 3, \ldots \).

a) If \( n \geq k \) are positive integers, show that

\[
|x_{n+1} - x_k| < c^k |x_1 - x_0|.
\]

(Hint: First bound \( |x_{n+1} - x_n| \) in terms of \( |x_1 - x_0| \).

b) Prove that the sequence \( \{x_n\} \) converges to a real number.
3. a) In the polynomial ring $\mathbb{Q}[x]$, consider the ideal $I$ generated by $x^4 - 1$ and $x^3 - x$. Does $I$ have a generator $f(x) \in \mathbb{Q}[x]$? Either find one or explain why none exists.

b) In the polynomial ring $\mathbb{Q}[x, y]$, do the same for the ideal generated by the polynomials $x$ and $y$. 
4. For each of the following, give either a proof or a counterexample.
   
a) Let $f$ be a continuous real-valued function on the open interval $0 < x < 3$. Must $f$ be uniformly continuous on the open interval $1 < x < 2$?

b) Suppose instead that $f$ is only assumed to be continuous on the open interval $0 < x < 2$. Must $f$ be uniformly continuous on the open interval $1 < x < 2$?
5. Let $V, W$ be two-dimensional real vector spaces, and let $f_1, \ldots, f_5$ be linear transformations from $V$ to $W$. Show that there exist real numbers $a_1, \ldots, a_5$, not all zero, such that $a_1 f_1 + \cdots + a_5 f_5$ is the zero transformation.
6. Evaluate $\int_C \left( e^{x^2} + \sin(y^2) \right)dx + \left( 2xy \cos(y^2) + xy^3 \right)dy$, where $C$ is the triangle with vertices (0, 0), (1, −1), (1, 1), oriented counterclockwise.
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7. Let \( f : X \to Y \) be a continuous map between metric spaces. For each of the following, give either a proof or a counterexample, using just the definition of compactness.

a) If \( A \subseteq X \) is compact, then so is \( f(A) \subseteq Y \).

b) If \( B \subseteq Y \) is compact, then so is \( f^{-1}(B) \subseteq X \).
8. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a constant $A$ such that

$$\int_0^x f(t)(1 + t^2)dt = \cos(x^2) + A.$$
9. For every integer \( n > 1 \), let \( U_n \) be the group of invertible elements of \( \mathbb{Z}/n\mathbb{Z} \) under multiplication.

a) Find the orders of \( U_8 \) and \( U_9 \). Explain.

b) Determine whether the groups \( U_8 \) and \( U_9 \) are cyclic.
10. Let \( f(x, y) = x^2 - xy + y^2 - y \).
   a) Does the function \( f \) achieve an absolute maximum on \( \mathbb{R}^2 \)? an absolute minimum on \( \mathbb{R}^2 \)? If so, find all points where this occurs.

b) Do the same with \( \mathbb{R}^2 \) replaced by the square \( 0 \leq x, y \leq 1 \).
11. Let $a, b, c$ be real numbers, and consider the matrix $A = \begin{pmatrix} a & b & c \\ b & c & b \\ c & b & a \end{pmatrix}$.

a) Explain why all the eigenvalues of $A$ must be real.

b) Show that some eigenvalue $\lambda$ of $A$ has the property that for every vector $v \in \mathbb{R}^3$, $v \cdot Av \leq \lambda \|v\|^2$. (Note: You are not being asked to compute the eigenvalues of $A$.)
12. Consider the differential equation \( y^{(4)} - y = ce^{2x} \) where \( c \) is a real constant.

a) Let \( S_c \) be the set of solutions of this equation. For which \( c \) is this set a vector space? Why?

b) For each such \( c \), find this solution space explicitly, and find a basis for it.