Problem 1. (a) Find all solutions in integers to the equation \(129x + 291y = 1\)
(b) Do the same for the equation \(129x + 291y = 3\)
Justify your assertions.

Solution. (a) There are no solutions. Indeed, since 3 \(\mid\) 129 and 3 \(\mid\) 291 we have that 3 \(\mid\) 129x + 291y, however 3 \(\nmid\) 1.

(b) Suppose we have a solution, i.e. a pair \((x_0, y_0)\) that satisfies 43\(x_0\) + 97\(y_0\) = 1. Then any other solutions is of the form \(x = x_0 + 97m, y = y_0 - 43m\) for \(m \in \mathbb{Z}\). Indeed, suppose \((x_1, y_1)\) is another solution then subtracting the two equations we obtain
\[
43(x_1 - x_0) + 97(y_1 - y_0) = 0 \tag{1}
\]
Since \(\gcd(43, 97) = 1\), taking equation (1) modulo 97 and 43 we find that \(x_1 - x_0 = 97m_1\) and \(y_1 - y_0 = 43m_2\). Plugging these two expressions into (1), we get \(43 \cdot 97m_1 + 43 \cdot 97m_2 = 0\), hence \(m_2 = -m_1\).

Finally we need to determine a special solution. By a variation of the Euclid’s algorithm we find that \((-9, 4)\) is a special solution. Therefore, the general solution is given by \(s_m = (-9 + 97m, 4 - 43m)\).

Problem 2. Show that \(f(x) = x^2\) is not uniformly continuous as a function on the whole real line (i.e. show for some \(\epsilon > 0\) there is no \(\delta > 0\) so that \(|f(x) - f(y)| < \epsilon\) whenever \(|x - y| < \delta\)).

Solution. Fix \(\epsilon > 0\) and \(\delta > 0\). It suffices to show that there are \(x, y\) such that \(|x - y| < \delta\) and \(|f(x) - f(y)| > \epsilon\). To that end, let \(x = \frac{1}{2}\epsilon + \frac{\delta}{2}\) and \(y = \frac{1}{2}\epsilon\). So,
\[
|f(x) - f(y)| = \frac{\delta}{2}(\frac{\epsilon}{2} + \frac{\delta}{2})
\[
> \frac{\delta}{2} \cdot \frac{\epsilon}{2}
\[
= \epsilon
\]

Problem 3. For each of the following, either give an example or explain why none exists.
(a) A non-abelian group of order 20.
(b) Two non-isomorphic abelian groups of order 30.
(c) A finite field whose non-zero elements form a cyclic group of order 17 under multiplication.
(d) A non-trivial automorphism of a finite field.
Solution. (a) There are non-abelian groups of order 20; one example is the dihedral group of 20 elements. This is the group of symmetries of a regular 10-agon, and, it is usually, denoted by $D_{10}$ or $D_{20}$.

(b) This is impossible. Since from the fundamental theorem of finitely generated abelian groups, a group with 30 elements is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(c) There is no such field. All finite fields have cardinality $p^n$ where $p$ is a prime. The multiplicative group of a field is cyclic, and has cardinality $p^n - 1$. Therefore, such a field must satisfy $p^n - 1 = 17$, or $p^n = 18$, which is impossible.

(d) There are groups with non trivial automorphisms. Take a field $F$ of characteristic $p$, with $p^n$ number of elements, where $n > 1$. Take $\phi : F \to F$ given by $a \mapsto a^p$. We show that $\phi$ is an automorphism. First, using the binomial expansion we see that $\phi(a + b) = \phi(a) + \phi(b)$, and of course $\phi(ab) = \phi(a)\phi(b)$, hence $\phi$ is a homomorphism. As $\ker \phi = \{0\}$, and $F$ is finite, $\phi$ is an isomorphism. We claim that $\phi$ is not trivial. Indeed the multiplicative group of a field is cyclic, with order $p^n - 1$. Hence, there is an $a \in F$ such that $a^p \neq a$. \hfill \blacksquare

Problem 4. Let $f$ be a real-valued continuous function defined for all $0 \leq x \leq 1$, such that $f(0) = 1$, $f(1/2) = 2$ ad $f(1) = 3$. Show that

$$\lim_{n \to \infty} \int_0^1 f(x^n) \, dx$$

exists and compute this limit. Justify your assertions.

Solution. The limit is equal to $\int_0^1 f(0) \, dx = 1$. Let $\epsilon > 0$. We find $\delta > 0$ such that $|f(x) - f(0)| < \frac{\delta}{2}$ for all $x \in [0, \delta]$. Now, pick $\delta_1 > 0$ and $N$ such that $\int_{1-\delta_1}^1 \max_{x \in [0,1]} |f(x) - 1| \, dx < \frac{\delta}{2}$ and $(1 - \delta_1)^n < \delta$ for all $n \geq N$.

Then for $n \geq N$ we get the following

$$\int_0^1 |f(x^n) - 1| \, dx = \int_0^{1-\delta_1} |f(x^n) - 1| \, dx + \int_{1-\delta_1}^1 |f(x^n) - 1| \, dx$$

$$\leq \int_0^{1-\delta_1} \frac{\epsilon}{2} \, dx + \int_{1-\delta_1}^1 \max_{x \in [0,1]} |f(x) - 1|$$

$$= (1 - \delta_1) \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$< \epsilon$$

And since $\left| \int_0^1 f(x^n) \, dx - 1 \right| = \left| \int_0^1 (f(x^n) - 1) \, dx \right| \leq \int_0^1 |f(x^n) - 1| \, dx$ we conclude. \hfill \blacksquare

Problem 5. Let $V$ be the real vector space consisting of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most 5 (including the 0 polynomial).

(a) Find a basis for $V$, and determine the dimension of $V$.

(b) Define $T : V \to \mathbb{R}^6$ by $T(f) = (f(0), f(1), f(2), f(3), f(4), f(5))$. Show that $T$ is a linear transformation and find its kernel.

(c) Deduce that for every choice of $a_0, a_1, \ldots, a_5 \in \mathbb{R}$ there is a unique polynomial $f(x) \in \mathbb{R}[x]$ of degree at most 5 such that $f(j) = a_j$ for $j = 0, 1, \ldots, 5$. 

2
Solution. (a) We have the following description for $V = \{a_6x^6 + a_5x^5 + \cdots + a_1x + a_0 | a_i \in \mathbb{R}, \text{for} 1 \leq i \leq 6\}$, the canonical basis is $e_i = x^i$. From the definition of $V$ we gave, $V = \langle e_1, e_2, \ldots, e_6 \rangle$. To see why $e_i$ are linearly independent take $\lambda_i$ for $1 \leq i \leq 6$ such that

$$
\lambda_1e_1 + \lambda_2e_2 + \cdots + \lambda_6e_6 = 0. 
$$

The polynomial $p(x)$ defined at the LSH of equation (2), must be the zero polynomial since only the zero polynomial has infinite many roots. The coefficients of $p(x)$ are exactly the scalars $\lambda_i$, therefore $\lambda_i = 0$ for all $i$, which in turns establishes that $e_i$ are linearly independent.

(b) The operator $T$ is linear, indeed,

$$
T(\lambda f + \mu g) = ((\lambda f + \mu g)(0), (\lambda f + \mu g)(1), \ldots, (\lambda f + \mu g)(5))
$$

$$
= (\lambda f(0) + \mu g(0), \lambda f(1) + \mu g(1), \ldots, \lambda f(5) + \mu g(5))
$$

$$
= \lambda T(f) + \mu T(g)
$$

The kernel of $T$ is trivial. Indeed, suppose $T(f) = 0$ then $f = 0$ as any non-constant polynomial of degree at most 5 has at most 5 roots.

(c) Suppose $f, g$ are two polynomials in $V$ such that $f(j) = g(j) = a_j$ for all $j = 1, 2, \cdots, 5$. We can express the previous statement via the operator $T$ as $T(f) = T(g)$ which in turn implies $T(f - g) = 0$, therefore $f = g$. ■

Problem 6. (a) Is there a metric space structure on the set $\mathbb{Z}$ such that the open sets are precisely the subsets $S \subset \mathbb{Z}$ such that $\mathbb{Z} - S$ is finite, and also the empty set?
(b) Is there a metric space structure on the set $\mathbb{Z}$ such that every subset is open?
Justify your assertions.

Solution. (a) No. All metric structures are Hausdorff, however the topology at hand is not. A topology is Hausdorff if for every two points $x, y$ there are open sets $V_x, V_y$ such that $x \in V_x, y \in V_y$ and $V_x \cap V_y = \emptyset$. To see why the topology is not Hausdorff, notice that for $V$ a non-trivial open set there is a $M \in \mathbb{Z}$ such that $\{M, M+1, \cdots, M+n, \cdots\} \subset V$, hence any two open sets (non-empty) $V, U$ intersect non-trivially.

(b) Yes. The discrete metric $d$, defined by $d(x, y) = 1$ if $x \neq y$ and zero otherwise, induces a topology such that every subset is open. To see this note that the singletons are open, and recall that union of open sets is open. ■

Problem 7. Let $\vec{F}$ be a vector field defined in $\mathbb{R}^3$ minus the origin defined by

$$
\vec{F}(\vec{r}) = \frac{\vec{r}}{||\vec{r}||^3} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}
$$

for $\vec{r} \neq 0$.
(a) Compute $\text{div} \, \vec{F}$.
(b) Let $S$ be the sphere of radius 1 centered at $(x, y, z) = (2, 0, 0)$. Compute

$$
\iint_S \vec{F} \cdot \vec{\nu} \, dS.
$$
Solution. (a) By definition of the divergence operator we have
\[
\text{div} \vec{F} = \frac{\partial(x/||\vec{r}||^3)}{\partial x} + \frac{\partial(y/||\vec{r}||^3)}{\partial y} + \frac{\partial(z/||\vec{r}||^3)}{\partial z}
\]
\[
= 3\left(\frac{x^2}{||\vec{r}||^5} - \frac{y^2}{||\vec{r}||^5} - \frac{z^2}{||\vec{r}||^5}\right)
\]
\[
= 0
\]

(b) Since the singular point of the vector field is not inside Conv(S) the convex hull of S, we can apply the divergence theorem.
\[
\iiint_S \vec{F} \cdot \vec{n} \, dS = \iiint_{\text{Conv}(S)} \text{div} \vec{F} \, dV
\]
\[
= \iiint_{\text{Conv}(S)} 0 \, dV
\]
\[
= 0
\]  

Problem 8. Let \( \{a_n\} \) be a bounded sequence of real numbers. Consider the infinite series
\[
f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n}
\]
where \( x \) is a real number. Prove that for any \( c > 1 \) this series converges uniformly on \( \{x \in \mathbb{R} | x \geq c\} \).

Solution. Define the power series \( p(x) = \sum_{n=1}^{\infty} a_n x^n \). To find \( R \), its radius of convergence, we calculate \( \limsup \sqrt[n]{|a_n|} \). The sequence \( a_n \) is bounded; so there is \( M > 0 \) such that \( |a_n| < M \), and, since \( \sqrt[n]{M} \to 1 \), we conclude that \( \limsup \sqrt[n]{|a_n|} \leq 1 \). So, \( R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \geq 1 \).

For every \( \delta > 0 \), a power series with radius of convergence \( R \) converges uniformly on \((R-\delta,R+\delta)\). Therefore, for any \( c \) such that \( 0 < \frac{1}{c} < 1 \), \( p(x) \) converges uniformly on \( A = (0, \frac{1}{c}] \). Since \( f(1/x) = p(x) \), \( f \) converges uniformly on \( \frac{1}{x}((0, \frac{1}{c}]) = [c, \infty) \) as desired. \( \blacksquare \)

Problem 9. Let \( A \) be the ring of continuous functions \( f : \mathbb{R} \to \mathbb{R} \), under (pointwise) addition and multiplication.

(a) Determine whether \( A \) is an integral domain.

(b) Let \( I \subset A \) be the subset consisting of functions \( f \) such that \( f(0) = 0 \). Is \( I \) an ideal?

What is \( A/I \)?

Solution. (a) The ring \( A \) is not an integral domain. Indeed, take \( f^+(x) = x \cdot 1_{[0,\infty)}(x) \) and \( f^-(x) = x \cdot 1_{(-\infty,0]}(x) \), then \( f^+ \cdot f^- = 0 \).

(b) Yes, as the \( I \) has additive subgroup structure, since \( (f-g)(0) = 0 \) for all \( f, g \in I \); and the multiplication is absorbing, i.e. \( (rf)(0) = r(0) \cdot 0 = 0 \) for all continuous functions \( r \). The ring \( A/I \) is isomorphic to \( \mathbb{R} \), to see this define \( \phi : A/I \to \mathbb{R} \) where \( \phi([f]) = f(0) \). First, the map \( \phi \) is well-defined since \( [f] = [g] \iff f(0) = g(0) \). Furthermore, \( \phi \) is a ring homomorphism since \( \phi([fg]) = f(0)g(0) = \phi([f])\phi([g]), \phi([f+g]) = f(0) + g(0) = \phi([f]) + \phi([g]). \)
We show that \( \phi \) is a bijection. For surjectivity notice that any constant map \( c \) is continuous. Now, to prove that \( \phi \) is injective we calculate its kernel: \( \phi([f]) = 0 \iff f(0) = 0 \iff f \in I \), therefore \( \ker \phi = \{[0]\} \). Hence, indeed, \( \phi \) is a ring isomorphism.

Problem 10. Suppose \( \{a_n : n = 1, n = 2, \ldots\} \) is a sequence of real numbers so that

\[
\sum_{n=1}^{\infty} |a_n| = 1.
\]

Let \( f(x) \) be given by the cos series

\[
f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).
\]

Prove that the series for \( f \) converges and that \( f \) is continuous.

Solution. Define \( S_m = \sum_{n=1}^{m-1} a_n \cos(nx) \). Firstly, notice that \( f \) exists since the series \( \sum_{n=1}^{\infty} a_n \cos(nx) \) is absolutely convergent. To show that \( f \) is continuous, it suffices to show that \( S_m \) converges uniformly to \( f \), since uniform convergence preserves continuity. Let \( \epsilon > 0 \), and choose \( N \) such that \( \sum_{n=N}^{\infty} |a_n| < \epsilon \). Then, for all \( m \geq N \)

\[
|f(x) - S_m(x)| = \left| \sum_{n=m}^{\infty} a_n \cos(nx) \right| \\
\leq \sum_{n=m}^{\infty} |a_n| |\cos(nx)| \\
= \sum_{n=m}^{\infty} |a_n| \\
\leq \sum_{n=N}^{\infty} |a_n| < \epsilon
\]

Taking sup over all \( x \) establishes the uniform convergence.

Problem 11. Let

\[
M = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(a) Find the minimal and characteristic polynomial of \( M \).
(b) Is \( M \) similar to a diagonal matrix \( D \) over \( \mathbb{R} \)? If so, find such a \( D \).
(c) Repeat part (b) with \( \mathbb{R} \) replaced by \( \mathbb{C} \) and also by the field \( \mathbb{Z}/5\mathbb{Z} \)

Solution. We calculate the characteristic polynomial \( \chi_M(x) \) of \( M \) by using the Laplace expansion,

\[
\begin{vmatrix}
-\lambda & 0 & 0 & 1 \\
1 & -\lambda & 0 & 0 \\
0 & 1 & -\lambda & 0 \\
0 & 0 & 1 & -\lambda
\end{vmatrix} = -1 \cdot \begin{vmatrix}
1 & -\lambda & 0 \\
0 & 1 & -\lambda \\
0 & 0 & 1
\end{vmatrix} - \lambda \cdot \begin{vmatrix}
1 & -\lambda & 0 \\
0 & 1 & -\lambda \\
0 & 0 & 1
\end{vmatrix} = \lambda^4 - 1
\]
So, \( \chi_M(\lambda) = \lambda^4 - 1 \). To find the \( \mu_M \) the minimal polynomial of \( M \) we distinguish two cases. First assume that the underlying field is not of characteristic 2. In this case \( \gcd(X'M, XM) = 1 \) which shows that \( X'M \) does not have double roots, therefore \( \mu_M = \chi_M \). If the characteristic is 2, then \( \chi_M(\lambda) = (\lambda - 1)^4 \). Since, \( (M - I)^3 \neq 0 \) we conclude, again, that \( \chi_M = \mu_M \).

(b) A matrix over a field \( F \) is diagonalizable if and only its minimal polynomial in \( F \) splits in \( F \) and has distinct roots. Here, \( \mu_M(\lambda) \), since it has complex roots, does not split in \( \mathbb{R} \), hence \( M \) is not diagonalizable.

(c) In part (a) we established that \( \mu_M \) has distinct roots over both \( \mathbb{C} \) and \( \mathbb{Z}/5\mathbb{Z} \). So in order to determine whether \( M \) is diagonalizable we need to determine whether \( \mu_M \) splits. In \( \mathbb{C} \) every polynomial splits, so \( M \) is diagonalizable. Finding the roots of \( \mu_M \) yields

\[
D_C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i \\
\end{bmatrix}.
\]

Suppose \( F = \mathbb{Z}/5\mathbb{Z} \), since \( U_5 \), \( F \)'s multiplicative group, has order 4 we obtain that \( \mu_M(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) \). Therefore,

\[
D_F = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4 \\
\end{bmatrix}
\]

\[\blacksquare\]

**Problem 12.** Let \( V \) be the vector space of \( C^\infty \) real-valued functions on \( \mathbb{R} \). Consider the following maps \( T_i : V \to V \).

\[
T_1(f) = f'' - 6f' + 9f \\
T_2(f) = f' - xf \\
T_3(f) = ff'
\]

(a) Which of the maps \( T_i \) are linear transformations?

(b) For each one that is, find a basis for the kernel.

**Solution.** (a) First note, that \( D : C^\infty \to C^\infty \), defined by \( D(f) = f' \) is linear. Therefore, the operators \( T_1 \) and \( T_2 \) are linear as a linear sum of linear operators.

The operator \( T_3 \) is not a linear operator. Indeed, take \( f = x \). Then, \( T_3(2f) = 4x \) and \( 2T_3(f) = 2x \), hence \( T_3(2f) \neq 2T_3(f) \).

(b) To find the kernel for \( T_1 \) we need to solve the homogeneous ODE

\[y'' - 6y' + 9y = 0.\]

The characteristic polynomial is \( r^2 - 6r + 9 = (r - 3)^2 \). Therefore, a basis for the kernel is \( e^{3x}, xe^{3x} \).

For \( T_2 \) we have the ODE

\[y' - xy = 0.\]
The integrating factor is $M(x) = e^{-x^2}$, therefore the general solution is given by $y = ce^{\frac{x^2}{2}}$. So, we can pick $v = e^{\frac{x^2}{2}}$ as the basis vector.