DEFORMATIONS OF G_2-STRUCTURES, STRING DUALITIES AND FLAT HIGGS BUNDLES

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"L'hydre-Univers tordant son corps écaillé d'astres."

Victor Hugo, *Les Contemplations, VI.26*

"Meu incêndio é uma metamorfose

e a minha metamorfose é a madeira de um incêndio."

ABSTRACT

DEFORMATIONS OF G_2-STRUCTURES, STRING DUALITIES AND FLAT HIGGS BUNDLES

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We study M-theory compactifications on G_2-orbifolds and their resolutions given by total spaces of coassociative ALE-fibrations over a compact flat Riemannian 3-manifold Q. The flatness condition allows an explicit description of the deformation space of closed G_2-structures, and hence also the moduli space of supersymmetric vacua: it is modeled by flat sections of a bundle of Brieskorn-Grothendieck resolutions over Q. Moreover, when instanton corrections are neglected, we also have an explicit description of the moduli space for the dual type IIA string compactification. The two moduli spaces are shown to be isomorphic for an important example involving A_1-singularities, and the result is conjectured to hold in generality. We also discuss an interpretation of the IIA moduli space in terms of "flat Higgs bundles" on Q and explain how it suggests a new approach to SYZ mirror symmetry, while also providing a description of G_2-structures in terms of B-branes. The net result is two algebro-geometric descriptions of the moduli space of complexified G_2-structures $\mathcal{M}_{G_2}^{\mathbb{C}}$: one as a character variety and a mirror description in terms of a Hilbert scheme of points. Usual G_2-deformations are parametrized by spectral

covers of flat Higgs bundles.

We also discuss a few ongoing developments: (1) we propose a heterotic dual to our main example, (2) we explain how the moduli space of flat Higgs bundles fits into a family of moduli spaces of extended Bogomolnyi monopoles, and (3) we introduce a natural variation of Hodge structures over $\mathcal{M}^{\mathbb{C}}_{G_2}$, and conjecture this space admits the structure of a complex integrable system.

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Chapter 1

Introduction

The main idea developed in this thesis is that a certain duality in String theory admits a geometric realization relating deformations of G_2-structures φ on a G_2-space M to deformations of flat connections on a "dual" Calabi-Yau manifold X. This translates the original differential-geometric problem into a more tractable deformation problem in algebraic geometry.

The duality works as follows: first one complexifies the moduli space of G_2-structures $\mathcal{M}_{G_2}$, obtaining a Kähler space $\mathcal{M}_{G_2}^{\mathbb{C}}$. *M-theory/IIA duality* predicts that if M admits a $U(1)$-action fixing an associative submanifold L, then $\mathcal{M}_{G_2}^{\mathbb{C}}$ must be isomorphic to the "IIA moduli space" $\mathcal{M}_{IIA}$ of $X := M/U(1)$. This moduli space parametrizes deformations of the complexified Kähler structure on X and deformations of certain geometric objects called *A-branes* on X.

We focus on a local model where M is given by a fibration of ADE singularities over a 3-manifold Q. In this situation, the space of deformations of the complexified G_2-structure $\varphi_{\mathbb{C}}$ admits a special subspace given by fiberwise hyperkähler deformations. Classically, the duality states that such subspace is mapped to the moduli space $\mathcal{M}_{\mathrm{IIA}}$ of A-branes wrapping the zero-section on $X := T^*Q$. Thus, the problem becomes giving an appropriate mathematical description of A-branes. This is obtained by performing a dimensional reduction of the supersymmetry condition on X down to its zero-section Q. The condition on X is the existence of a *Hermitian-Yang-Mills connection.* We show that the dimensional reduction gives a flat complex connection $\mathcal{A}$ on a vector bundle $E \to Q$. This translates deformations of $\varphi_{\mathbb{C}}$ into deformations of $\mathcal{A}$.

However, mathematically one is interested in deformations of φ, not $\varphi_{\mathbb{C}}$. Thus, an important problem is to identify exactly which deformations of $\mathcal{A}$ describe deformations of φ. There are two key ingredients in order to achieve this: the first is the *Corlette-Donaldson theorem*, which in our setup implies (Theorem 5.1.8 below) that the data $(E, \mathcal{A})$ is equivalent to (a generalization of) what we call a *flat Higgs bundle* (E, A, θ, h) on Q. Here, h is a harmonic metric on E, which always exists when $\mathcal{A}$ is sufficiently nice. Moreover, $\mathcal{A} = A + \theta$, where A is a flat unitary connection ($D_A h = 0$) and θ is a *flat Higgs field* ($D_A\theta = 0$, $\theta \wedge \theta = 0$, $\theta + \theta^\dagger = 0$). This provides a canonical decomposition of the deformations of $\varphi_{\mathbb{C}}$ into real deformations parametrized by A, and imaginary deformations parametrized by θ. The second ingredient is Theorem 5.3.1, the *Spectral Correspondence for flat Higgs bundles*: Higgs data (E, A, θ, h) can be translated into *spectral data* $(S_Q, L, a, \widetilde{h})$ where $S_Q \to Q$ is a finite branched cover, $L \to S_Q$ is a line bundle and a is a flat $U(1)$-connection on L $(\nabla_a \widetilde{h} = 0)$.

Having this setup in mind, we focus on G_2-*platyfolds*, which we define as ADE-bundles over a compact *flat* 3-manifold Q (such a Q is called a *platycosm*). There are two reasons for this choice: the first one is that flat compact manifolds are finite quotients of tori, so their character varieties (i.e., moduli space of flat connections) map to the character variety of a torus - also known as the moduli space of commuting triples - a well-understood space. The second reason is that using the flatness condition we are able to build a deformation family of closed G_2-structures parametrized exactly by the spectral covers of flat Higgs bundles.

There are 10 affine isomorphism classes of platycosms, and we prove that only one of them allows G_2-orbifolds fitting the framework of M-theory/IIA duality[1]. This space is called the *Hantzsche-Wendt manifold*, and following convention we denote it by $\mathcal{G}_6$. We analyze in detail the duality for the total space of an ADE-fibration over $\mathcal{G}_6$ with McKay group $\mathbb{Z}_2$. In particular, we compute the character variety $\mathrm{Char}(\mathcal{G}_6, SL(2, \mathbb{C}))$ of $\mathcal{G}_6$ and we check that it agrees with the moduli space $\mathcal{M}^{\mathbb{C}}_{G_2}$ associated to this specific G_2-orbifold.

Chapters 2 3, 4 and 5 comprise the main body of the thesis. They are organized as follows: we start chapter 2 with a discussion of flat geometry and the classification

[1]More precisely, we prove that only one such manifold allows a $\mathcal{N} = 1$ compactification.

of the platycosms; we then briefly discuss character varieties, a topic that will be recurrent in the thesis. In chapter 3, we define ADE G_2-platyfolds and define a deformation family for closed G_2-structures coming from hyperkähler deformations of its fibers. This is the most technical (and arguably the most interesting) result in this thesis. In the final sections we discuss $\mathcal{N} = 1$ compactifications and study the calibrated submanifolds of $\mathcal{G}_6 \times_{\mathbb{K}} \mathbb{C}^2/\Gamma$, the Hantzsche-Wendt G_2-platyfold first considered in Acharya's work [Ach98]. In chapter 4 we describe the type IIA Calabi-Yau dual of this G_2-geometry in full detail: the SYZ fibration and special Lagrangian deformations, and the character varieties describing the moduli space of A-branes. In chapter 5 we define flat Higgs bundles over a three-manifold and establish a spectral construction. This result ties the deformation problems from the previous chapters together. In chapter 6 we propose a SYZ mirror for our IIA geometry and describe its moduli space of B-branes as a Hilbert scheme of points. Finally, in chapter 7 we explore a few future directions related to this work: in section 7.1 we study the heterotic dual of our G_2-geometry and its moduli space; section 7.3 relates flat Higgs bundles to Kapustin-Witten systems; and section 7.2 studies a variation of Hodge structure over $\mathcal{M}^{\mathbb{C}}_{G_2}$ inspired by the Hodge-theoretic formulation of conifold transitions [DDP07], [DDP06] and the connection between Large N duality and the G_2-flop [AV01]. We also propose an approach to build a complex integrable system over $\mathcal{M}^{\mathbb{C}}_{G_2}$.

Notation: Throughout this thesis, we denote by $\pi : E \to N$ a fiber bundle, fibration, or more generally a family of spaces over N; and denote the total space of the bundle or family simply by E. G_c is a complex semisimple Lie group, $\mathfrak{g}_c$ is its Lie algebra, $\mathfrak{h}_c$ is a Cartan subalgebra, and W the Weyl group. We denote by G the compact real form of G_c, and $\mathfrak{g}$, $\mathfrak{h}$ are the associated compact real Lie algebra and Cartan subalgebra. A connection on a G-bundle $E \to N$ is an element $A \in \Omega^1_N(Ad_G(E))$. We denote its horizontal distribution by $\mathbf{H}_A \subset TE$ and the covariant derivative by ∇_A. Relative differential k-forms on $E \to N$ are denoted by $\Omega^k(E/N)$. Unless stated otherwise, all manifolds we work with are connected and without boundary. Whenever we speak of the fundamental group of a manifold, we will assume a base-point has been fixed once and for all and we will suppress it from the notation. Finally, we use two different notations for the fixed set of the action of a group K on a space Y: either $\mathrm{Fix}(K)$ or Y^K, depending whether Y is understood from context or not.

We assume the reader is familiar with the basics of G_2-geometry. Standard references for this subject are Joyce's books [Joy00], [Joy07] and Hitchin's paper [Hit00]. Knowledge of the theory of Higgs bundles and spectral covers is desirable but not entirely necessary. Good references on this topic are [Hit87], [Hit87a], [Sim92], [Don95] [DM95], [Sch12].

Chapter 2

Flat Riemannian geometry

2.1 Flat Riemannian manifolds

The G_2-spaces we will study are total spaces of bundles over compact, flat Riemannian 3-manifolds, so we start with a review of flat Riemannian geometry. The results in this section are used heavily in the main body of the text. For a more complete introduction, we refer to the books by Charlap [Cha86] and Szczepański [Scz12].

Definition 2.1.1. Let $\mathrm{Iso}(\mathbb{R}^n) := O(n) \ltimes \mathbb{R}^n$ be the group of rigid motions on $\mathbb{R}^n$. A subgroup $\pi \leqslant \mathrm{Iso}(\mathbb{R}^n)$ is called *crystallographic* if it is a discrete subgroup acting on $\mathbb{R}^n$ such that $\mathbb{R}^n/\pi$ is compact. It is called *torsion-free* if the action is free.

Definition 2.1.2. A subgroup $\pi \leqslant \mathrm{Iso}(\mathbb{R}^n)$ is called *Bieberbach* if it acts properly discontinuously on $\mathbb{R}^n$ in such a way that $\mathbb{R}^n/\pi$ is compact.

The second definition is equivalent to $\mathbb{R}^n/\pi$ being a compact flat manifold. We call such a space a *Bieberbach manifold*. Note that π is crystallographic if and only if $\mathcal{G} := \mathbb{R}^n/\pi$ is a compact flat orbifold (a *Bieberbach space*). It is clear that π is Bieberbach if and only if it is a torsion-free crystallographic group.

Any crystallographic group π fits into a short exact sequence

$$0 \to \Lambda \to \pi \to H \to 1 \tag{2.1.1}$$

where H_π is a finite group we call the *monodromy* of π and Λ is a free abelian H-module ($\cong \mathbb{Z}^n$ as a group). So it is classified by an element ς of the group cohomology $H^2(H, \Lambda)$. This description follows from a theorem of Zassenhaus: a subgroup $\pi \leqslant Iso(\mathbb{R}^n)$ is crystallographic if and only if it has a normal, maximal abelian, free abelian subgroup $\mathbb{Z}^n$ of finite index.

The most important result in this subject is *Bieberbach's theorem:*

Theorem 2.1.3. *(Bieberbach): Let $\pi \subset Iso(\mathbb{R}^n)$ be a crystallographic group, and $Q^n := \mathbb{R}^n/\pi$ the associated Bieberbach space. Let $\boldsymbol{T}^n$ be a flat n-torus.*

1. *The monodromy H is finite and the pure translations $\Lambda := \pi \cap \mathbb{R}^n$ of π form a lattice.*

   Equivalently, there is a finite normal covering map $\boldsymbol{T}^n \to Q^n$ which is a local isometry.

2. *Every isomorphism between crystallographic subgroups of $Iso(\mathbb{R}^n)$ is given by*

a conjugation in $\mathit{Aff}(\mathbb{R}^n)$.

Equivalently, two Bieberbach spaces of the same dimension and with isomorphic fundamental groups are affinely isomorphic.

3. *There are only finitely many isomorphism classes of crystallographic subgroups of $\mathit{Iso}(\mathbb{R}^n)$.*

   Equivalently, there are finitely many affine classes of Bieberbach spaces of dimension n.

We note that part 3 essentially follows from the fact that number of exact sequences 2.1.1 is bounded by the order of the finite group $H^2(H, \Lambda)$.

Remark 2.1.4. In the context of flat geometry, the terms "holonomy" and "monodromy" are essentially interchangeable. Accordingly, we may write H as either H_π or H_{Q^n} to emphasize that it is the *monodromy* group of π or the *holonomy* group of the Bieberbach space Q^n. This is consistent with standard terminology, as H is the holonomy of the flat metric on Q^n inducing the monodromy action on the fundamental group π.

Clearly, $\mathbb{R}^n$ is the universal cover of Q^n, and $\pi_1(Q^n) = \pi$. The first part of Bieberbach's theorem implies that the H-action on $\Lambda \cong \pi_1(\mathbf{T}^n)$ is induced from a free H-action on $\mathbf{T}^n$ such that $Q \cong \mathbf{T}^n/H$. It is clear that $\mathbf{T}^n$ is also a Bieberbach manifold, with trivial monodromy. For this reason, we call $\mathbf{T}^n$ the *monodromy cover* of Q^n. The existence of the monodromy cover strongly constrains the possible holonomies of Bieberbach manifolds (see Proposition 2.1.7 for the classification in three dimensions). This is in stark contrast with the theory for *non-compact* flat Riemannian manifolds: it is a theorem of Auslander and Kuranishi that every finite group is the holonomy group of some flat manifold.

We also have the following useful result:

Theorem 2.1.5. *(Charlap, Hiss-Szczepański): Let π be a crystallographic group fitting into the exact sequence*

$$0 \to \Lambda \to \pi \to H_\pi \to 1 \tag{2.1.2}$$

Let $\varsigma \in H^2(H_\pi, \Lambda)$ be the group cohomology element classifying 2.1.2. The following are equivalent:

1. *π is a Bieberbach group*
2. *For any injection $\iota : \mathbb{Z}_p \hookrightarrow H_\pi$, $\iota^*\varsigma \neq 0$*
3. *For each $q \in Q^n$, the holonomy representation $h_q : H_\pi \to T_q Q^n$ is reducible*

We refer to the components of the orthogonal representation of H_π as *isotypic components*.

2.1.1 Platycosms

Definition 2.1.6. A three-dimensional Bieberbach manifold is called a *platycosm* [CR03].

The name literally means "flat universe" and is based on the idea that such spaces are alternative geometries to the (almost) flat three-space we live in.[1]

Proposition 2.1.7. *There are* 10 *affine equivalence classes of platycosms,* 6 *of which are orientable. They are classified by their holonomy groups as follows:*

- $\mathcal{G}_1$ *is the flat three-torus* $\boldsymbol{T}$*, so the holonomy is trivial:* $H_{\mathcal{G}_1} = \{1\}$
- $\mathcal{G}_2$ *with* $H_{\mathcal{G}_2} \cong \mathbb{Z}_2$
- $\mathcal{G}_3$ *with* $H_{\mathcal{G}_3} \cong \mathbb{Z}_3$
- $\mathcal{G}_4$ *with* $H_{\mathcal{G}_4} \cong \mathbb{Z}_4$
- $\mathcal{G}_5$ *with* $H_{\mathcal{G}_5} \cong \mathbb{Z}_6$
- $\mathcal{G}_6$ *with* $H_{\mathcal{G}_6} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

We will follow this notation, except for $\mathcal{G}_1$ which we will just denote by $\mathbf{T}$.

The space $\mathcal{G}_6$ will be particularly important for us. It is known in the literature as the *Hantzsche-Wendt manifold* or the *didicosm.* Explicit descriptions for $H_{\mathcal{G}_6}$ and $\Lambda_{\mathcal{G}_6}$ are:

$$H_{\mathcal{G}_6} = \left\langle A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\rangle \subset SO(3) \tag{2.1.3}$$

[1] See [AL14] for an analysis of the topology of our large-scale spatial universe using cosmic microwave background radiation, and the suitability of platycosm models.

$$\Lambda_{\mathcal{G}_6} = \left\langle \left(A, \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}\right), \left(B, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}\right) \right\rangle \subset SO(3) \ltimes \mathbb{R}^3 = \mathrm{Iso}^+(\mathbb{R}^3) \tag{2.1.4}$$

2.2 Character Varieties of Bieberbach groups

In this section we prove a result that will be used repeatedly in what follows. Let G be any group. The exact sequence

$$1 \to \Lambda \to \pi \xrightarrow{q} H \to 1$$

induces another sequence:

$$1 \to \mathrm{Hom}(H, G) \to \mathrm{Hom}(\pi, G) \xrightarrow{\bar{r}} \mathrm{Hom}(\Lambda, G) \tag{2.2.1}$$

For any group A define the *character variety* $\mathrm{Char}(A, G)$ to be the GIT quotient of $\mathrm{Hom}(A, G)$ by the conjugation action of G. We have induced maps of character varieties:

$$\mathrm{Char}(H, G) \to \mathrm{Char}(\pi, G) \xrightarrow{r} \mathrm{Char}(\Lambda, G) \tag{2.2.2}$$

For $h \in H$ let C_h denote the conjugation map by h, and let H act on $Hom(\Lambda, G)$ by

$$h(\rho) = \rho \circ C_{\tilde{h}} \quad \forall h \in H \tag{2.2.3}$$

where $\tilde{h} \in \pi$ is such that $q(\tilde{h}) = h$. It is easy to see that if $q(\tilde{h}_1) = q(\tilde{h}_2)$, then because Λ is abelian, $C_{\tilde{h}_1} = C_{\tilde{h}_2}$ and hence the action is well-defined. Moreover, the action descends to an action of H on $\mathrm{Char}(\Lambda, G)$ in the obvious way.[2] Let $\mathrm{Fix}(H)$ denote the subset of $\mathrm{Char}(\Lambda, G)$ consisting of elements fixed by H. The next lemma states that $r\big(\mathrm{Char}(\pi, G)\big) = \mathrm{Fix}(H)$.

Lemma 2.2.1. *Suppose $\rho \in Hom(\Lambda, G)$ is such that $\rho = \overline{r}(\tilde{\rho}) = \tilde{\rho}|_\Lambda$ for $\tilde{\rho} \in Hom(\pi, G)$. Then $[\rho] \in Fix(H)$.*

Conversely, if $C_G(\rho(\Lambda)) = 0$ and $[\rho] \in Fix(H)$, then $\exists [\tilde{\rho}] \in Char(\pi, G)$ such that $r([\tilde{\rho}]) = [\rho]$.

Proof. Let $h \in H$. Then:

$$\begin{aligned} h(\rho) &= h(\tilde{\rho}|_\Lambda) \\ &= \tilde{\rho}(\tilde{h}) \circ \tilde{\rho}|_\Lambda \circ \tilde{\rho}(\tilde{h})^{-1} \\ &= C_{\tilde{\rho}(\tilde{h})}(\tilde{\rho}|_\Lambda) \\ &= C_{\tilde{\rho}(\tilde{h})}(\rho) \end{aligned}$$

hence $h[\rho] = [\rho]$, i.e. $[\rho] \in \mathrm{Fix}(H)$.

Conversely, $h[\rho] = [\rho] \implies \rho \circ C_{\tilde{h}} = S_{\tilde{h}} \rho S_{\tilde{h}}^{-1}$ for some $S_{\tilde{h}} \in G$. It is easy to see that if $a \in \mathrm{Ker}(q)$, then $S_a^{-1}\rho(a) \in C_G(\rho(\Lambda))$. Hence $S_a = \rho(a)$. Define $\tilde{\rho} : \pi \to G$

[2]Note that since it is an action by an outer conjugation of Λ, it descends non-trivially to the quotient.

by $\tilde{\rho}(x) = S_x$, $\forall x \in \pi$. Then clearly $\tilde{\rho}|_{\mathrm{Ker}(q)} = \rho$ and if $x, y \in \pi$, the hypothesis on the centralizer implies that $S_{xy} = S_x S_y$, so $\tilde{\rho}(xy) = \tilde{\rho}(x)\tilde{\rho}(y)$. So $\tilde{\rho} \in \mathrm{Hom}(\pi, G)$ with $r([\tilde{\rho}]) = [\rho]$.

□

2.2.1 A-branes wrapping platycosms

Later on we will be interested in studying a certain space: the moduli space of A-branes wrapping a platycosm Q. We will show these are classified by flat G-bundles on Q, where G is a simply-connected Lie group. A choice of flat connection on a G-vector bundle over Q corresponds to a point of the character variety/stack:

$$\mathrm{Char}(Q, G) := \mathrm{Hom}\big(\pi_1(Q), G\big) \big/ \mathcal{C}_G \tag{2.2.4}$$

where $\mathcal{C}_G$ denotes the conjugation action in G. The quotient is more correctly taken in the stacky sense, but to simplify matters we will restrict to the GIT quotient. From the previous section, we know that this space is essentially determined by the action of H on the character variety of the monodromy cover $\mathbf{T}$.

We now describe the character varieties of tori up to dimension 3. Let $\mathcal{T}$ be a maximal torus for G.

For $\mathbf{S}^1$ the problem is trivial: the generator of $\mathbb{Z} = \pi_1(\mathbf{S}^1)$ can be mapped anywhere in G. Hence $\mathrm{Char}(\mathbf{S}^1, G) = G/\mathcal{C}_G \cong \mathcal{T}/W$.

For a two-torus $\mathbf{T}^2$, $\mathrm{Char}(\mathbf{T}^2, G)$ is given by two commuting elements in G up to conjugation. Let $g \in G$ and $h \in C_G(g)$, the centralizer of g. It is known that, for simply-connected G, the centralizer $C_G(g)$ is connected (Bott's theorem), so we can first conjugate g to $\mathcal{T}$ and then conjugate h to the torus of $C_G(g)$, which by connectedness is just $\mathcal{T}$. The net result is that g and h can be simultaneously conjugated to lie on the maximal torus $\mathcal{T}$. The maximal tori are conjugated by elements of the Weyl group W. Hence the character variety is:

$$\mathrm{Char}(\mathbf{T}^2, G) = \mathcal{T} \times \mathcal{T} / W \tag{2.2.5}$$

For a three-torus, $\mathrm{Char}(\mathbf{T}^3, G)$ is now given by three commuting elements modulo conjugation. So now we need to determine all possible configurations of $g, h, k \in G$, with $g \in \mathcal{T}$ and $h, k \in C_G(g)$, i.e., the *moduli space of commuting triples.* This problem was solved by Borel, Friedman and Morgan [BFM02] and Kac and Smilga [KS99], who showed that if the classification of commuting triples is essentially determined by the fundamental group of the centralizers. Commuting triples (g, h, k) whose semi-simple part of the centralizers is simply-connected can always be conjugated to the maximal torus, giving one of the components of the moduli space:

$$\mathcal{T} \times \mathcal{T} \times \mathcal{T} / W \tag{2.2.6}$$

However, there are also *non-trivial* commuting triples. This happens when G has elements whose semisimple part of the centralizer has torsion. These extra commuting triples produce new connected components in the character variety.

Essentially, torsion in $\pi_1(C_g(G))$ occurs when the root system of $\mathfrak{h}$ admits non-trivial *coroot integers*. Each divisor of a coroot integer is called a *level* ℓ, and each ℓ determines a subtorus $\mathcal{T}(\ell)$ of $\mathcal{T}$ given by the intersection of the kernels of the roots whose coroot integers are *not* divisible by ℓ. The torus $\mathcal{T}(\ell)$ has an associated Weyl group $W_{\mathcal{T}(\ell)} := N_G(\mathcal{T}(\ell))/C_G(\mathcal{T}(\ell))$.

Each ℓ determines $\phi(\ell)$ connected components for the character variety, where ϕ is Euler's totient function; each connected component is given by:

$$\mathcal{T}(\ell) \times \mathcal{T}(\ell) \times \mathcal{T}(\ell) \Big/ W_{\mathcal{T}(\ell)} \tag{2.2.7}$$

In particular, for $G = SL(n, \mathbb{C})$, the only allowed level is $\ell = 1$ and there are no non-trivial commuting triples. Thus:

$$\mathrm{Char}\big(\mathbf{T}^3, SL(n, \mathbb{C})\big) \;=\; \big((\mathbb{C}^*)^{n-1}\big)^3 \Big/ \Sigma_n \tag{2.2.8}$$

and similarly:

$$\mathrm{Char}\big(\mathbf{T}^3, SU(n)\big) \;=\; \big(U(1)^{n-1}\big)^3 \Big/ \Sigma_n \tag{2.2.9}$$

Chapter 3

Deformations of G_2-orbifolds

3.1 ADE G_2-platyfolds

We start by fixing the following data:

1. Q is an oriented platycosm, δ its flat Levi-Civita connection and $\pi := \pi_1(Q)$ the associated Bieberbach group

2. $\mathbf{V} \to Q$ a rank one *quaternionic* vector bundle (i.e., the structure group is $Sp(1) \leqslant SL(2, \mathbb{C})$)

3. Γ a finite subgroup of $Sp(1)$, and hence a fiberwise action of Γ on $\mathbf{V}$

4. A flat quaternionic connection ∇ on $\mathbf{V} \to Q$ compatible with the Γ-action in an appropriate sense (see remark below)

5. A flat volume form $\mu \in \Omega^3(Q, \mathbb{R})$

Remark 3.1.1. 1. A flat connection ∇ compatible with Γ is given by an action of $\pi \times \Gamma$ on $\widetilde{Q} \times \mathbb{H}$, where $\widetilde{Q}$ is the universal cover of Q and Γ acts trivially on $\widetilde{Q}$. Equivalently, we have an action of π on $\mathbb{H}$ commuting with the Γ-action. This is the same as a representation of π on the centralizer $C_{Sp(1)}(\Gamma) \leqslant Sp(1)$, i.e., the conjugacy class of an element of $\mathrm{Hom}(\pi, C_{Sp(1)}(\Gamma))$. The trivial homomorphism gives rise to the trivial flat connection (i.e., with no monodromy).

2. This data fixes a "flat fiberwise quaternionic structure", i.e., a tri-section (I, J, K) of $Aut_{\mathbb{H}}(\mathbf{V}) \to Q$ such that $\nabla I = \nabla J = \nabla K = 0$.

In the language of Goldman's geometric structures [Gol88], δ defines a torsion-free $(\mathbb{R}^3, \mathrm{Iso}^+(\mathbb{R}^3))$-structure on Q with graph TQ, and $(\mathbf{V}, \nabla)$ is the graph of a $(\mathbb{R}^4, Sp(1))$-structure on Q. This last structure is then required to be compatible with the group Γ. We will require these two geometric structures to interact in a specific way when we discuss G_2-deformations.

Definition 3.1.2. We call (Γ, ∇) *ADE data* for $\mathbf{V}$.

The first thing we need to determine is, for a fixed Γ, when does $\mathbf{V}$ admit *nontrivial ADE data*, i.e., when $\mathrm{Hom}(\pi, C_{Sp(1)}(\Gamma)) \neq 0$ modulo conjugation. This is a compatibility condition between the topology of Q and the Γ-compatible $(\mathbb{R}^4, Sp(1))$-structure. We now show that its existence depends (up to one exception) only on the ADE type of Γ.

Proposition 3.1.3. *Nontrivial ADE data for* $\mathbf{V}$ *exists for all platycosms with cyclic holonomy, and for* $\mathcal{G}_6$ *when* Γ *is of type* A_n.

Proof. The centralizer depends on the ADE type of Γ. Here are the possibilities:

- Γ *of type* A_n: there are two subcases. If $n = 1$, then $\Gamma \cong \mathbb{Z}_2$ and

$$C_{Sp(1)}(\mathbb{Z}_2) = Sp(1) \tag{3.1.1}$$

  If $n \geqslant 2$, then $\Gamma \cong \mathbb{Z}_n$ and Γ lies on a maximal torus $\mathcal{T}$ of $Sp(1) \cong SU(2)$. The centralizer is just the torus itself:

$$C_{Sp(1)}(\mathbb{Z}_n) = \mathcal{T} \cong U(1) \tag{3.1.2}$$

- Γ *of type* D_n *for* $n > 2$, E_6, E_7 *or* E_8: Then:

$$C_{Sp(1)}(\Gamma) = Z(Sp(1)) \cong \mathbb{Z}_2 \tag{3.1.3}$$

Note that $Hom(\mathbb{Z}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2^3$, while $Hom(\mathbb{Z}^3, U(1)) \cong U(1)^3$. It is also true that $Hom(\mathbb{Z}^3, Sp(1)) \cong U(1)^3$, as one can always conjugate three commuting elements to a maximal torus of $Sp(1) \cong SU(2)$ [BFM02] [KS99]. It follows that ADE data exists for $\mathbf{T} = \mathcal{G}_1$ irrespectively of Γ. Now, for any group G, we obtain from 2.1.2 an exact sequence:

$$1 \to \mathrm{Hom}(H_\pi, G) \to \mathrm{Hom}(\pi, G) \to \mathrm{Hom}(\mathbb{Z}^3, G) \tag{3.1.4}$$

Representations that are conjugate in G are considered isomorphic, so we are interested in the image of r in:

$$\mathrm{Char}(H_\pi, G) \to \mathrm{Char}(\pi, G) \xrightarrow{r} \mathrm{Char}(\mathbb{Z}^3, G) \tag{3.1.5}$$

where $\mathrm{Char}(A, G) := \mathrm{Hom}(A, G)/\mathcal{C}_G$ is the *Character Variety of* A and will be studied in more detail in Chapter 4. What we need to know right now is that the action of H_π on $\mathbb{Z}^3$ induces an action on $\mathrm{Hom}(\mathbb{Z}^3, G)$ that descends to $\mathrm{Char}(\mathbb{Z}^3, G)$. From Lemma 2.2.1, the image of r is given by the fixed set of this action.

For platycosms with cyclic holonomy the monodromy action fixes a direction in $\mathbb{R}^3$, which implies the descendant action on the character variety has non-trivial fixed points. This implies that nontrivial ADE data can be chosen in those cases.

In the case when $Q = \mathcal{G}_6$, simple inspection determines that the action of $H_{\mathcal{G}_6}$ on $\mathbb{R}^3$ has no fixed points, so the previous argument does not apply. The argument in this case requires a careful examination of $Im(r)$ in 3.1.5, which depends on the ADE type of G. The proof that non-trivial A_n data (i.e., when $G = SL(n, \mathbb{C})$) can be chosen for $\mathcal{G}_6$ will be a consequence of our computation of $\mathrm{Char}(\pi_1(\mathcal{G}_6), SL(n, \mathbb{C}))$ in Chapter 4.

□

This has the following consequences for the structure of the bundle $\mathbf{V}$:

- If Γ is of type A_1, any flat connection on $\mathbf{V}$ is compatible with Γ.

  If Γ is of type A_n for $n \geqslant 2$, then the structure group reduces to $U(1) \leqslant Sp(1)$ and $\mathbf{V} \cong \mathbf{L} \oplus \mathbf{L}^{-1}$, where $\mathbf{L}$ is a flat complex line bundle.

- If Γ is of type D_n for $n \geqslant 3$ or of types E_6, E_7 or E_8, then $\mathbf{V} \cong \mathbf{L} \oplus \mathbf{L}$, where $\mathbf{L}$ is a flat complex line bundle such that $\mathbf{L}^{\otimes 2}$ is the trivial complex line bundle.

Given data $(Q, \mathbf{V}, \Gamma, \mathbf{L}, \nabla, I, J, K, \mu)$ as above, the quaternionic structure (I, J, K) determines a triple $\underline{\omega}_0 := (\omega_I, \omega_J, \omega_K)$ of fiberwise hyperkähler structures. The integrability condition implies that $\nabla\omega_I = \nabla\omega_J = \nabla\omega_K = 0$. Our next goal is to understand under which circumstances the data $(Q, \mathbf{V}, \Gamma, \mathbf{L}, \nabla, I, J, K, \mu)$ induces a closed G_2-structure on $\mathbf{V}$ such that $\mathbf{V} \to Q$ is a coassociative fibration. We start with some examples.

Example 3.1.4. $\mathbf{V} = \mathbb{C}^2 \times \mathbf{T}$ has a standard closed G_2-structure:

$$\varphi = \sum_{i=1}^{3} dx_i \wedge \omega_i + dx_{123} \tag{3.1.6}$$

for a choice of flat coordinates $\{x_i\}$ on $\mathbb{T}^3$ and hyperkähler structure $\underline{\omega}$ on $\mathbb{C}^2$.[1] Here and in what follows, we use the notation $dx_{123} := dx_1 \wedge dx_2 \wedge dx_3$.

Note that because there is no monodromy, the local section dx_i glues to a global flat section, so the formula makes sense globally. We think of $\mathbf{V}$ as the total space of the trivial flat vector bundle $\mathbf{V} \to \mathbf{T}$. It is easy to check that $\varphi|_{\mathbb{C}^2} = 0$ and $\star\varphi|_{\mathbb{T}^3} = 0$, so the fibers $\mathbb{C}^2$ are coassociative and the zero-section $\mathbb{T}^3$ is associative. In fact, this G_2-structure is also torsion-free. Its associated metric is just the flat metric, which of course has holonomy $\{1\} \subset G_2$.

Up to a change of basis, $\underline{\omega}$ is a $SU(2)$-triple, and since $\Gamma \leqslant SU(2)$, $\underline{\omega}$ can be

[1]More generally, $\underline{\omega}$ can be a *hypersymplectic structure* - see defintion 3.1.12 below.

taken to be Γ-invariant. Thus there is a well-defined G_2-structure on the quotient $M = \mathbb{C}^2/\Gamma \times \mathbb{T}^3$ with the same properties.

Example 3.1.5. Consider $\mathbb{C}^2 \times \mathcal{G}_6$. Even though this is trivial as a smooth bundle, there is monodromy from the flat metric connection δ, so we do not wish to consider it as a trivial *flat* bundle; the formula

$$\varphi = \sum_{i=1}^{3} dx_i \wedge \omega_i + dx_{123} \tag{3.1.7}$$

can still be written down, but only on local patches $U_\lambda \subset \mathcal{G}_6$ belonging to a flat trivialization $\mathfrak{U} = \{U_\lambda; \lambda \in \Lambda\}$ of $T\mathcal{G}_6$. The monodromy transformations for the dx_i's on $U_{\lambda\lambda'} := U_\lambda \cap U_{\lambda'}$ are given by the action of $H_{\mathcal{G}_6}$ given by the matrices A, B, AB in 2.1.3. If the ω_i's are chosen such that $\mathbb{K} := H_{\mathcal{G}_6}$ acts by the *inverses* $A^{-1}, B^{-1}, (AB)^{-1}$ on the local patches[2], then the element:

$$\eta := \sum_{i=1}^{3} dx_i \wedge \omega_i \tag{3.1.8}$$

glues to a global flat section. Obviously dx_{123} also glues globally, so together they give a well-defined G_2-structure.

The question is then: can such ω_i's be chosen? To induce the correct action, one needs to pick a non-trivial flat bundle $\mathbf{V} := \mathbb{C}^2 \times_{\mathbb{K}} \mathcal{G}_6$. The index $\mathbb{K}$ refers to the holonomy of the flat structure: i.e., we need to choose an element of $\rho \in$

[2]For $\mathcal{G}_6$ the inverses actually coincide with the original matrices, but in general this is not the case.

$\mathrm{Hom}(\pi_1(\mathcal{G}_6), Sp(1))$ with $\rho(\pi_1(\mathcal{G}_6)) \cong \mathbb{K}$. The monodromy group $\mathbb{K}$ then acts on the sheaf of triples of vertical 2-forms $\Omega^2(\mathbb{C}^2)^3$ over $\mathcal{G}_6$. The inverse action above can be written as $(\omega_1, \omega_2, \omega_3) \mapsto (\pm\omega_1, \pm\omega_2, \pm\omega_3)$ (where we have exactly 2 minus signs), i.e. it is an action by hyperkähler rotations.[3] The conclusion is then that once a local hyperkähler triple is chosen, if one changes it by the appropriate hyperkähler rotations on local patches, one gets a global closed G_2-structure. In this case, one can also check that the G_2-structure is torsion-free, and the associated metric has holonomy $\mathbb{K} \subset G_2$.

Example 3.1.6. This example first appeared in [Ach98]. Take $\mathbf{V} = \mathbb{C}^2 \times_{\mathbb{K}} \mathcal{G}_6$ as in the previous example and let $\Gamma \cong \mathbb{Z}_2 \leqslant Sp(1)$ act on $\mathbb{C}^2$ in the natural way. It is easy to see that this action is compatible with the $\mathbb{K}$-action: this means that the monodromy representation ρ of $\mathbf{V}$ is an element of $\mathrm{Hom}(\pi_1(\mathcal{G}_6), C_{Sp(1)}(\mathbb{Z}_2))$, which is clear since $C_{Sp(1)}(\mathbb{Z}_2) = Sp(1)$. It follows that the previous example descends to a closed, torsion-free G_2-structure on $M := \mathbf{V}/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6$ and on the resolution $\widehat{M} := \widetilde{\mathbb{C}^2/\mathbb{Z}_2} \times_{\mathbb{K}} \mathcal{G}_6$. In this last space, the associated metric has holonomy $SU(2) \rtimes \mathbb{K} \subset G_2$.

Note that if one takes $\Gamma = \mathbb{Z}_n$, then $C_{Sp(1)}(\mathbb{Z}_n) = U(1)$ does not contain $\mathbb{K}$. In this situation the singularity $\mathbb{C}^2/\mathbb{Z}_n$ acquires non-trivial monodromy dictated by $[\mathbb{K}, \mathbb{Z}_n] \subset Sp(1)$.

Example 3.1.7. This example shows that picking the action on the hyperkähler triple

[3] Notice this would not work if we allowed *non-orientable* platycosms.

to be given by the inverse monodromy matrices is not always the correct choice. Let $Q = \mathcal{G}_3$ and consider $\mathbb{C}^2 \times \mathcal{G}_3$. Choose again local flat 1-forms dx's. The holonomy $H_{\mathcal{G}_3} \cong \mathbb{Z}_3$ is generated by the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{3.1.9}$$

The correct action of $H_{\mathcal{G}_3}$ on $\Omega^2(\mathbb{C}^2)^3$ is not given by A^{-1}, but by $(A \circ R_3)^{-1}$ where R_3 is reflection on the xy-plane. In other words, the correct matrix is obtained by reflecting the lower 2×2-block on its anti-diagonal:

$$(A \circ R_3)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \tag{3.1.10}$$

Again this is just a hyperkähler rotation on local intersections, and as before it defines a global flat 3-form η on $\mathbf{V} = \mathbb{C}^2 \times_{\mathbb{Z}_3} \mathcal{G}_3$ such that $\eta + dx_{123}$ is a closed G_2-structure.

We can now do the same thing we did for the previous example: pick a finite subgroup $\Gamma \cong \mathbb{Z}_n \leqslant Sp(1)$ and note that $C_{Sp(1)}(\mathbb{Z}_n) = U(1)$ always contains a $\mathbb{Z}_3$. Hence the flat bundle $\mathbf{V}$ can be taken to be compatible with the Γ-action on $\mathbb{C}^2$ and we get well-defined closed, torsion-free G_2-structures on $\mathbb{C}^2/\mathbb{Z}_n \times_{\mathbb{Z}_3} \mathcal{G}_3$ and $\widetilde{\mathbb{C}^2/\mathbb{Z}_n} \times_{\mathbb{Z}_3} \mathcal{G}_3$. The metric on the last space has holonomy $SU(2) \rtimes \mathbb{Z}_3$.

Examples involving the platycosms $\mathcal{G}_2, \mathcal{G}_4$ and $\mathcal{G}_5$ are similar to the $\mathcal{G}_3$ example, the only essential difference being that in the absence of $\mathbb{Z}_3$ factors in the monodromy group, one can also work with singularities of types D and E. On the other hand, this makes the $\mathcal{G}_6$ example much more interesting, and the main reason we focus on it in further chapters: $H_{\mathcal{G}_6} \cong \mathbb{K}$ is the only platycosm holonomy group such that $SU(2) \rtimes H_Q$ cannot be conjugated to a subgroup of $SU(3) \subset G_2$ (see section 3.3). This feature implies that the manifold $\widetilde{M}$ above, which has holonomy $SU(2) \rtimes \mathbb{K}$, defines an appropriate compactification for M-theory.

Assume one is given $(\mathbf{V}, \nabla) \to \mathbf{T}$ a nontrivial flat bundle. Now there is no monodromy coming from $\mathbf{T}$, but the flat connection ∇ has an associated monodromy group H_∇, which we assume is a finite subgroup of $Sp(1)$. In this situation, one needs to reverse the argument: the dx_i's are to be chosen to be compatible with the ω_i's on a flat trivialization of $\mathbf{V}$. This is because the hyperkähler condition imposes that the ω_i's transform according to the action of H_∇ on local patches. The action is defined by a choice up to conjugation of element in $\mathrm{Hom}(\pi_1(\mathbf{T}), C_{Sp(1)}(\Gamma))$, which can be seen [BFM02] as a choice of an element in $U(1)^3/\mathbb{Z}_2$, possibly with restrictions depending on Γ. In any case, the ω_i's transform via three commuting hyperkähler rotations, and we just need to define local sections dx_i's on each patch that are related via the inverse transformations on the intersections. This guarantees that η is globally defined, and since everything happens in $SO(3)$, dx_{123} is also globally defined.

In order to obtain the correct action on the dx_i's, one needs to define an appropriate action of H_∇ on $\mathbf{T}$. Hence, we need to consider $(\mathbf{V}, \nabla)$ as a flat bundle over the quotient space $\mathbf{T}/H_\nabla$. If the action of H_∇ on $\mathbf{T}$ is taken to be free and properly discontinuous, the quotient space must be one of the 10 platycosm. The constraints on H_∇ imposed by the classification 2.1.7 seems to be in contradiction with our freedom in choosing H_∇ to be any ADE subgroup. However, upon closer inspection, one sees that the short exact sequence 2.1.1 is not unique; one can modify the lattice Λ, for example by modifying its period along one direction, as long as one modifies the group H_π accordingly.

Example 3.1.8. To illustrate this last point, start with the exact sequence for $\mathcal{G}_6$:

$$1 \to \mathbb{Z}^3 \to \pi_1(\mathcal{G}_6) \to \mathbb{K} \to 1 \tag{3.1.11}$$

Bieberbach's first theorem says that $\mathcal{G}_6$ is a quotient of the three-torus $\mathbf{T}$. This is realized via the following (free) action of $\mathbb{K} = \langle \alpha, \beta \rangle$ on $\mathbf{T}$:

$$\alpha(x_1, x_2, x_3) = (-x_1, -x_2 + \frac{1}{2}, x_3 + \frac{1}{2})$$
$$\beta(x_1, x_2, x_3) = (x_1 + \frac{1}{2}, -x_2, -x_3)$$

So $\mathcal{G}_6 = \mathbf{T}/\mathbb{K}$. However, a second possible description is $\mathcal{G}_6 = \mathbf{T}/D_8$, where $D_8 \cong \mathbb{Z}_2 \rtimes \mathbb{K}$ is the dihedral group with 8 elements. Let $D_8 = \langle \alpha', \beta' \rangle$. The action is given by:

$$\alpha'(x_1, x_2, x_3) = (-x_1, -x_2 + \frac{3}{4}, x_3 + \frac{1}{2})$$
$$\beta'(x_1, x_2, x_3) = (x_1 + \frac{1}{4}, -x_2 + \frac{1}{4}, -x_3)$$

This provides a second short exact sequence for $\pi_1(\mathcal{G}_6)$:

$$1 \to \Lambda \to \pi_1(\mathcal{G}_6) \to D_8 \to 1 \tag{3.1.12}$$

where the lattice Λ is given by $2\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Note that $(\beta')^2(x_1, x_2, x_3) = (x_1+\frac{1}{2}, x_2, x_3)$ is a translation by an order 2 element generating the center $\mathbb{Z}_2 \leqslant D_8$, and as such it doesn't contribute to the holonomy.

This new action is also compatible with $\Gamma = \mathbb{Z}_2$, so the resulting closed, torsion-free G_2-structure descends to $M = \mathbf{V}/\Gamma$. The only difference between this example and example 3.1.6 is that here there is an extra central $\mathbb{Z}_2$-symmetry acting on the hyperkähler triple. This symmetry is not visible at the geometric level, but it has to be remembered when using the string dualities explored in this paper. For example, the symmetry defined by $(\beta')^2$ gives rise to a so-called *B-field* on dual Calabi-Yau spaces. Mathematically, this is given by a flat $\mathbb{Z}_2$-gerbe defined on these Calabi-Yaus. We will leave a more detailed discussion on B-fields (and their relation to monodromy of ADE singularities) for future work.

Example 3.1.9. Up to now, in all examples it was possible to write down a closed G_2-structure on $\mathbf{V}$ that descends to M. Now suppose $\mathbf{V} \to Q$ has a flat connection

∇ with monodromy group H_∇ acting by a representation ρ_∇, and the platycosm Q has a nontrivial holonomy representation $\rho_Q : H_Q \to SO(3)$. Now both groups of local sections need to be chosen in a compatible way. First, we need to pick a common flat trivialization $\mathfrak{U}$ for ∇ and δ. On local intersections, we need $\rho_\nabla(\underline{\omega})$ to cancel out $\rho_Q(\underline{dx})$. In general H_∇ is isomorphic to a quotient of $\pi_1(Q)$, and even if it happens that $H_Q \leqslant H_\nabla$ and that the actions satisfy $(\rho_\nabla)|_{H_Q} = (\rho_Q)^{-1}$, there might still be other subgroups of H_∇ that act non-trivially on $\underline{\omega}$, which will spoil the gluing construction for η.

Thus, for a fixed Q, there are restrictions on which $(\mathbf{V}, \nabla)$ are allowed. If one chooses ∇ such that the lattice subgroup $\mathbb{Z}^3 \leqslant \pi$ acts trivially, then $H_\nabla = H_Q$ and we just need to impose the inversion condition. However, not all platycosms will admit flat bundles with this property; indeed, going back to the exact sequence

$$1 \to \mathrm{Hom}(H_Q, C_{Sp(1)}(\Gamma)) \to \mathrm{Hom}(\pi, C_{Sp(1)}(\Gamma)) \to \mathrm{Hom}(\mathbb{Z}^3, C_{Sp(1)}(\Gamma))$$

what we are looking for is a nontrivial element $\rho \in \mathrm{Hom}(\pi, C_{Sp(1)}(\Gamma))$ that maps to 0, i.e., we need a nontrivial element of $\mathrm{Hom}(H_Q, C_{Sp(1)}(\Gamma))$. These do not exist if Q is either $\mathcal{G}_3$ or $\mathcal{G}_5$ and Γ is of type D_n or $E_{6,7,8}$. For the other platycosms, at least one such element exists, since in this case all subgroups of H_Q have even order and one can pick the map sending all generators to $-\mathbf{1}$.

For type A_n, $C_{Sp(1)}(\Gamma)$ is big enough and one can always arrange such data for any platycosm.

We summarize these examples in the following:

Proposition 3.1.10. *Suppose Q is a platycosm and Γ an ADE group such that $(Q,\Gamma) \notin \{(\mathcal{G}_3, DE), (\mathcal{G}_5, DE)\}$. Then there is a nontrivial Γ-compatible quaternionic flat bundle $(\mathbf{V} \to Q, \nabla)$ with a closed G_2-structure $\overline{\varphi}_0$ that descends to a closed G_2-structure φ_0 on $M_0 := \mathbf{V}/\Gamma$.*

It is clear that M_0 is the total space of a bundle of ADE-singularities of type Γ over the platycosm Q. In particular, $Q \hookrightarrow M_0$ as the zero section and is a codimension-four orbifold singularity in M. This inspires the following definition.

Definition 3.1.11. Given data as above, we say that M_0 with its induced closed G_2-structure φ_0 is an *ADE G_2-platyfold of type* (∇, Γ).

We will often drop the reference to ∇ (and therefore to $\mathbf{V}$) if it is implicit in the discussion. ADE G_2-platyfolds and their "string duals" will be the main subject of this thesis.

Let $p : M \to Q$ be the ADE G_2-platyfold constructed above. There is an exact sequence:

$$0 \to Ker(dp) \to TM \stackrel{dp}{\to} TQ \to 0 \tag{3.1.13}$$

where $V := Ker(dp)$ is called the *vertical bundle*. A connection on M is equivalent to a section $s : TQ \hookrightarrow TM$, i.e., a splitting of the sequence; it defines a *horizontal distribution* $\mathbf{H} = s(TQ) \subset TM$. This induces a splitting of the exterior derivative on M into $d = d_f + d_h + F_{\mathbf{H}}$, where d_f is a fiberwise differential, d_h a horizontal differential and $F_{\mathbf{H}}$ is the curvature operator of $\mathbf{H}$. In our situation, the connection on M is induced from the flat connection ∇ on $\mathbf{V}$, so $F_\nabla = 0$.

In order to gain a better understanding of the integrability conditions on φ, it is useful to work in a slightly more general setup. Most of the discussion in the rest of this section follows [Don16] closely.

Definition 3.1.12. A *hypersymplectic structure* on an oriented four-manifold X is a triple $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ of symplectic forms such that at each point $p \in X$, $\underline{\omega}_p$ spans a maximal positive-definite subspace of $\Lambda^2(X)$ with respect to the wedge product.

In other words, $\omega_i \wedge \omega_j \in \Gamma(X, \mathrm{Sym}^2(X))$ has positive determinant at every point, and by rescaling the volume form one can take $\det(\omega_i \wedge \omega_j) = 1$ at all points. Thus $G_{ij} := \omega_i \wedge \omega_j$ is a Riemannian metric, and it is hyperkähler if and only if $\omega_i \wedge \omega_j$ is a multiple of the identity.

Accordingly, we define a *hypersymplectic element* on $(M \to Q, \nabla)$ to be an element $\eta \in \mathbf{H}^* \oplus \Lambda^2 V^*$ such that at each point q, the linear map $\eta_q : \mathbf{H}_q \to \Lambda^2 V_q^*$ injects $\mathbf{H}_q$ as a maximal positive subspace with respect to the wedge product.

We have the following theorems of Donaldson [Don16]:

Theorem 3.1.13. *(Donaldson): A closed G_2-structure on $(M \to Q, \mathbf{H})$ with coassociative fibers and orientation compatible with those of M and Q is equivalent to a choice of the following data:*

- *A hypersymplectic element $\eta \in \mathbf{H}^* \oplus \Lambda^2 V^*$ satisfying:*

$$d_h \eta = 0$$

$$d_f \eta = 0$$

- *A tensor* $\mu \in \Lambda^3 \mathbf{H}^*$ *satisfying:*

$$d_h \mu = 0$$

$$d_f \mu = -F_{\mathbf{H}}(\eta)$$

which is pointwise positive when seen as an element of $\Lambda^3 T^* Q \cong \underline{\mathbb{R}}$.

Theorem 3.1.14. *(Donaldson) A closed* G_2*-structure as in Theorem 3.1.13 is torsion-free if and only if the following holds:*

$$d_f \gamma = -F_{\mathbf{H}} \nu$$

$$d_h \gamma = 0$$

$$d_f \nu = 0$$

$$d_h \nu = 0$$

where γ *and* ν *are determined from* η *and* μ *by:*

$$\gamma_i \wedge \eta_j = \delta_{ij} \big(\mu \det(\eta \wedge \eta)\big)^{1/3}$$

$$\nu = \det(\eta \wedge \eta)^{1/3} \mu^{-2/3}$$

We refer to $(\mathbf{H}, \eta, \mu)$ as *Donaldson data* for a G_2-structure on M. The G_2 3-form is given by $\varphi = \eta + \mu$, and its dual 4-form is $\psi = \gamma + \nu$. The equations for $(\mathbf{H}, \eta, \mu)$ in Theorem 3.1.13 will be called *Donaldson's constraints for a closed G_2-structure.*

A third theorem of Donaldson will also be important:

Theorem 3.1.15. *(Donaldson): Given a* hyperkähler *element η along the fibers of $M \to Q$ and a positive 3-form μ on Q, there is a unique connection $\mathbf{H}$ on $M \to Q$ such that $(\mathbf{H}, \eta, \mu)$ satisfies Donaldson's constraints for a closed G_2-structure on M.*

It is clear now why our examples provided closed G_2-structures: they were just special cases of Donaldson data, in situations where the connection is flat. This simplification gives an *a posteriori* reason to work with platycosms: while they have plenty of flat connections, their character varieties are quite simple and can be described explicitly. Deeper reasons will arise in the next section, where we will use Donaldson's theorems to study deformations of ADE G_2-platyfolds via unfolding of singularities; and in the next chapter, where the spectral data associated to a deformation will be described explicitly.

3.2 Deformation family for closed G_2-structures

3.2.1 The Kronheimer family

Recall that we denote by $\mathfrak{g}_c$ a semi-simple complex Lie algebra, $\mathfrak{h}_c$ a Cartan subalgebra, and W the Weyl group. The compact real form of $\mathfrak{g}_c$ is denoted $\mathfrak{g}$, and $\mathfrak{h}$ is the associated real Cartan subalgebra.

We start this section by reviewing the construction of the Brieskorn-Grothendieck versal deformation of the quotient singularity $\mathbb{C}^2/\Gamma$ via Slodowy slices: let x be a subregular nilpotent element of $\mathfrak{g}_c$ and complete it to a $\mathfrak{sl}(2,\mathbb{C})$-triple (x,h,y). Define the *Slodowy slice*:

$$\mathcal{S} = x + \mathfrak{z}_{\mathfrak{g}_c}(y) \subset \mathfrak{g}_c \tag{3.2.1}$$

where $\mathfrak{z}_{\mathfrak{g}_c}(y)$ is the centralizer of y, i.e. the kernel of the adjoint action of G_c on y.

Consider the GIT adjoint quotient $\mathfrak{g}_c \to \mathfrak{g}_c//G_c$. Chevalley's theorem says that $\mathbb{C}[\mathfrak{g}_c]^{G_c} \cong \mathbb{C}[\mathfrak{h}_c]^W$, so $\mathfrak{g}_c//G_c \cong \mathfrak{h}_c/W$. Define $\Psi : \mathcal{S} \to \mathfrak{h}_c/W$ to be the restriction of $\mathfrak{g}_c \to \mathfrak{h}_c/W$ to $\mathcal{S}$.

Theorem 3.2.1. *(Slodowy [Slo80]): The family Ψ has the following properties:*

1. *Ψ is a flat, surjective holomorphic map*

2. *$\Psi^{-1}(0) \cong \mathbb{C}^2/\Gamma$*

3. *Given any other map $\Psi' : \mathcal{A} \to \mathcal{B}$ satisfying properties* 1 *and* 2 *there is a map*

$\beta : (\mathcal{B}, b) \to (\mathfrak{h}_c/W, 0)$ such that $\Psi' = \beta^*\Psi$. The map β might not be unique, but its derivative $d\beta_b$ is unique.[4]

4. *Ψ is equivariant with respect to natural $\mathbb{C}^*$-actions on $\mathcal{S}$ and $\mathfrak{h}_c/W$*

In other words, Ψ is the *Brieskorn-Grothendieck $\mathbb{C}^*$-miniversal deformation of* $\mathbb{C}^2/\Gamma$. Thus, the Slodowy slice is a geometric realization of the deformations of $\mathbb{C}^2/\Gamma$ inside the Lie algebra $\mathfrak{g}_c$.

This embedding of $\mathcal{S}$ into $\mathfrak{g}_c$ comes with a symmetry group. Let $C = Z_{G_c}(x) \cap Z_{G_c}(y)$ be the *reductive centralizer* (of x with respect to h).[5] Its action on $\mathcal{S}$ commutes with $\mathbb{C}^*$, so there is an action of $\mathbb{C}^* \times C$ on $\mathcal{S}$. The action of C restricts to act on the fibers of Ψ (i.e., Ψ is C-invariant). The group $\mathbb{C}^* \times C$ is called the symmetry group of the Slodowy slice.

Lemma 3.2.2. *$C \cong \mathbb{C}^*$ for $\mathfrak{g}_c$ of type A_n, and $C = \{e\}$ for types D_n and $E_{6,7,8}$.*

Proof. See Slodowy's book [Slo80]. $\square$

Kronheimer [Kro89a] constructed a deformation space for ALE-structures on $\mathbb{C}^2/\Gamma$. He starts with (Y^k, I, J, K) a certain flat simply-connected hyperkähler space

[4] This uniqueness at the infinitesimal level is known as *miniversality*. Any two miniversal deformations of an ADE singularity are isomorphic, and their reduced Kodaira-Spencer map is an isomorphism.

[5] The name is due to the fact that the identity component C^0 is reductive, and the component groups $Z_{G_c}(x)/Z^0_{G_c}(x)$ and C/C^0 coincide.

with a quaternionic action of G (i.e., $G \leqslant Sp(k)$), and constructs a *hyperkähler moment map*:

$$\mu = (\mu_1, \mu_2, \mu_3) : Y^k \to \mathbb{R}^3 \otimes \mathfrak{g}^* \tag{3.2.2}$$

such that for $\xi \in \mathbb{R}^3 \otimes \mathfrak{h}^*$ the *hyperkähler quotient*:

$$S_\xi = \mu^{-1}(\xi)/G \tag{3.2.3}$$

is well-defined and is also a hyperkähler space. In particular, it has a hyperkähler triple $(I, J, K)_\xi$ induced from Y^k.

One can then prove that if $\xi \in \mathbb{R}^3 \otimes \mathfrak{h}$, S_ξ is an ALE space, and is non-singular if and only if $\xi \notin \bigcup_\upsilon \mathbb{R}^3 \otimes C_\upsilon$, where C_υ is the hyperplane orthogonal to a root υ.

Kronheimer's deformation family $\mathcal{K} \to \mathfrak{h}_c$ is constructed as follows: consider the *complexified moment map* $\mu_c := \mu_2 + i\mu_3 : Y^k \to \mathbb{C} \otimes \mathfrak{g}^*$, where $\mathbb{C} = (\{0\} \times \mathbb{R}^2, I)$, i.e., the complex structure I induces an identification $\mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{C}$ given by $(\chi_1, \chi_2, \chi_3) \mapsto (\chi_1, \chi_2 + i\chi_3)$. Then:

$$S_{(0,\chi_2,\chi_3)} = \big(\mu_1^{-1}(0) \cap \mu_c^{-1}(\chi_2 + i\chi_3)\big)/G \tag{3.2.4}$$

is an affine variety with respect to the complex structure $I_{(0,\chi_2,\chi_3)}$. After passing to a normalization, these spaces fit into the *Kronheimer deformation family:*

$$\Theta : \mathcal{K} \to \mathfrak{h}_c \tag{3.2.5}$$

which is a surjective flat holomorphic map with $\Theta^{-1}(0) \cong \mathbb{C}^2/\Gamma$.

Recall that Slodowy's family $\Psi : \mathcal{S} \to \mathfrak{h}_c/W$ is versal for $\mathbb{C}^2/\Gamma$, so Ψ must be induced from it by pullback from a map between the parameter spaces. Kronheimer proves that Θ is equivariant with respect to a $\mathbb{C}^*$-action on $\mathcal{K}$ and weight 2 dilations on $\mathfrak{h}_c$. Due to Looijenga's description of the period map for Ψ [Loo84], it follows that Θ is induced from Ψ via pullback by the projection map $p_W : \mathfrak{h}_c \to \mathfrak{h}_c/W$.

Definition 3.2.3. Fix non-zero $\chi_2, \chi_3 \in \mathfrak{h}$. We say $\chi_1 \in \mathfrak{h}$ is *generic* if $\xi = (\chi_1, \chi_2, \chi_3) \notin \mathbb{R}^3 \otimes C_\upsilon$ for any root υ.

If χ_1 is generic, the space S_ξ is a nonsingular hyperkähler manifold, and there is a *resolution of singularities* $r_\xi : S_\xi \to S_{(0,\chi_2,\chi_3)}$. Therefore, any appropriate choice of χ_1 induces a *simultaneous resolution*[6] $\widetilde{\Theta}_\xi : \widetilde{\mathcal{K}}_\xi \to \mathfrak{h}_c$ of Θ (i.e., $\widetilde{\Theta}_\xi = \Theta \circ r_\xi$).

We summarize Kronheimer's results in the following:

Theorem 3.2.4. *(Kronheimer): For every generic $\chi \in \mathfrak{h}$, there is a commutative diagram:*

[6]Tjurina [Tju70], building on previous work of Brieskorn [Bri68], proved that a flat holomorphic map $f : S \to T$ with two-dimensional fibers admitting at most finitely many rational double points admits a local resolution of singularities: around any point $t \in T$ there is an open set $U \subset T$ such that the family $f|_{f^{-1}(U)}$ admits a simultaneous resolution of all fibers, i.e. a commutative diagram whose maps restricted to the fibers are resolutions of singularities. Kronheimer's construction gives the simultaneous resolution for the Brieskorn-Grothendieck $\mathbb{C}^*$-miniversal deformation.

$$\begin{tikzcd}
\widetilde{\mathcal{K}_\chi} \arrow[r, "r_\chi"'] \arrow[d, "\tilde{\Theta}_\chi"'] & \mathcal{K} \arrow[r] \arrow[d, "\Theta"'] & \mathcal{S} \arrow[d, "\Psi"'] \\
\mathfrak{h}_c \arrow[r, "="] & \mathfrak{h}_c \arrow[r, "p_W"] & \mathfrak{h}_c/W
\end{tikzcd}$$

satisfying the following properties:

1. *$\widetilde{\Theta}_\chi$ is a flat, surjective holomorphic map, with fibers diffeomorphic to the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of $\mathbb{C}^2/\Gamma$*

2. *$\widetilde{\Theta}_\chi$ is a simultaneous resolution of Θ, i.e., $r_\chi|_{S_{(\chi,\chi_2,\chi_3)}}$ is a resolution of singularities of $S_{(0,\chi_2,\chi_3)}$*

3. *$\widetilde{\mathcal{K}}_\chi$ inherits a $\mathbb{C}^*$-action from Y^k such that $\widetilde{\Theta}_\chi$ is $\mathbb{C}^*$-equivariant*

Moreover, one can also prove:

Theorem 3.2.5. *(Kronheimer [Kro89]): Given a (smooth) hyperkähler ALE space S, there is a $\xi = (\chi, \chi_2, \chi_3)$ with χ generic such that $S \cong S_\xi$ as hyperkähler manifolds.*

3.2.2 A deformation family for hyperkähler structures

One should think of the base $\mathfrak{h}_c$ of the Kronheimer family $\widetilde{\mathcal{K}}_\chi$ as parametrizing *infinitesimal deformations of the holomorphic symplectic structure on* $\widetilde{\mathbb{C}^2/\Gamma}$. The reason is the following: let $\mathfrak{h}_>$ be the *positive Weyl chamber.* By the McKay correspondence, $\mathfrak{h}_>$ is isomorphic to the Kähler cone of $\widetilde{\mathbb{C}^2/\Gamma}$, with tangent spaces $\mathfrak{h}$. A choice of complex structure on $\widetilde{\mathbb{C}^2/\Gamma}$ induces an isomorphism $T(\mathfrak{h}_>)\otimes\mathbb{C} \cong \mathfrak{h}_c$, so the deformation parameter is a complexified Kähler class, which is in fact holomorphic [HKLR87]

For our purposes, we need to make a clear distinction between deformations of a *holomorphic symplectic structure* (HS) and deformations of a *hyperkähler structure* (HK). The main point is that, even though Kronheimer's construction produces all HK ALE spaces, it does not fit them together in a family induced from the Slodowy slice. In order to write diagram 3.2.4, one needs to fix a complex structure (say, I) and an element $\chi \in \mathfrak{h}$. This fixes the HK-structure but does not account for all deformations. However, we will need to work with the full HK family.

First we need to fix the complex structure. Let V be the adjoint representation of $SU(2)$. In comparing $\widetilde{\Theta}_{\chi'}$ and $\widetilde{\Theta}_{\chi}$, they correspond to different choices of Kähler classes for a fixed complex structure I inducing a linear isomorphism $V \cong \mathbb{R} \oplus \mathbb{C}$. In other words, the complex structure is fixed once a choice of splitting $V \cong \mathbb{R} \oplus \mathbb{C}$ has been made. We write $\mathfrak{h}_V := \mathfrak{h} \otimes V$.

Under the McKay identification $\mathfrak{h} \cong H^2(S_\xi, \mathbb{R})$, for every fixed $\xi = (\chi, \chi_2, \chi_3)$, one should think of the deformation parameter:

$$\chi_2 + i\chi_3 \in \mathfrak{h}_c \backslash \bigcup_{v \text{ root}} \mathbb{C} \otimes C_v \tag{3.2.6}$$

as a choice of cohomology class for a I-holomorphic symplectic form on the fiber $\widetilde{\Theta}^{-1}(\chi_2 + i\chi_3) \cong S_\xi$.

This is where the distinction between the HS and HK structures on the fibers comes in. For each $\chi \in \mathfrak{h}^\circ := \mathfrak{h} \backslash \bigcup C_v$, the family $\widetilde{\Theta}_\chi$ provides a HS-deformation of $\mathbb{C}^2/\Gamma$, meaning, a deformation of the singularity together with a two-form $\omega_c \in \Omega^2(\widetilde{\mathcal{K}}_\chi/\mathfrak{h}_c)$ that restricts to a a holomorphic symplectic form $\omega_c^{\chi_2+i\chi_3}$ on every fiber, varying holomorphically with $\chi_2+i\chi_3 \in \mathfrak{h}_c$. It is clear that for $\chi' \neq \chi$, the manifolds $\widetilde{\Theta}_{\chi'}$ and $\widetilde{\Theta}_\chi$ are isomorphic as holomorphic symplectic manifolds.

However, the associated *hyperkähler* manifolds are not the same. Indeed, the following well-known proposition shows that a HK-structure is equivalent to a HS-structure + a complex structure and a Kähler class:

Proposition 3.2.6. *(Beauville): Let (S, Ω) be a holomorphic symplectic manifold with a complex structure I and $[\omega] \in H^{1,1}(S)$ a Kähler class. Then there is a unique hyperkähler structure (I, J, K) on S such that $[\omega_I] = [\omega]$ and $\Omega = \omega_J + i\omega_K$.*

Proof. Follows from the Calabi-Yau theorem. □

Therefore, we work with the pullback of $\Theta : \mathcal{K} \to \mathfrak{h}_c$ by the projection map $p_I : \mathfrak{h}_V \to \mathfrak{h}_c$. We denote this family by $\Xi : \mathcal{Q} \to \mathfrak{h}_V$. The fibers are $\Xi^{-1}(\chi, \chi_2, \chi_3) = X_{(0,\chi_2,\chi_3)}$.

We now "glue" all families $\widetilde{\mathcal{K}}_\chi$ together and define a family $\widetilde{\Xi} : \widetilde{\mathcal{Q}} \to \mathfrak{h}_V$ and a map $\mathfrak{r} : \widetilde{\mathcal{Q}} \to \mathcal{Q}$ such that $(\widetilde{\Xi}, \mathfrak{r})|_{\{\chi\}\times\mathfrak{h}_c} = (\widetilde{\mathcal{K}}_\chi, r_\chi)$.

Proposition 3.2.7. *There is a family of spaces $\widetilde{\Xi} : \widetilde{\mathcal{Q}} \to \mathfrak{h}_V$ and a diagram:*

$$\begin{tikzcd} \widetilde{\mathcal{Q}} \arrow[r] \arrow[d, "\widetilde{\Xi}"'] & \mathcal{S} \arrow[d, "\Psi"] \\ \mathfrak{h}_V \arrow[r, "p_W \circ p_I"] & \mathfrak{h}_c/W \end{tikzcd}$$

Each generic fiber $\tilde{\tilde{\Xi}}^{-1}(\chi, \chi_2, \chi_3)$ is a hyperkähler deformation of $\widetilde{\mathbb{C}^2/\Gamma}$. Moreover, the hyperkähler triple on each fiber is induced from a relative triple $\underline{\omega}_{unf} \in \left(\Omega^2(\tilde{\mathcal{Q}}/\mathfrak{h}_V^\circ)\right)^3$ varying smoothly with $\mathfrak{h}_V^\circ$.

The notation $\underline{\omega}_{\mathrm{unf}}$ is meant to emphasize that this element induces HK-structures on the *unfoldings* S_ξ of the singularity $\mathbb{C}^2/\Gamma$.

Proof. By Kronheimer's construction, the element (χ_2, χ_3) determines the class of a I-holomorphic symplectic form $\omega_c^{\chi_2+i\chi_3}$ on the fiber $\tilde{\tilde{\Xi}}^{-1}(\chi, \chi_2, \chi_3)$. The choice of $\chi \in \mathfrak{h}$ determines a Kähler class ω^χ and hence a fixed hyperkähler structure. Under the identification $\mathfrak{h} \cong T\mathfrak{h}_>$, one can think of (χ, χ_2, χ_3) as a "tangent hyperkähler vector" on the Kähler cone of $\widetilde{\mathbb{C}^2/\Gamma}$. The global relative triple is defined by $\underline{\omega}_{\mathrm{unf}}(\chi, \chi_2, \chi_3) = (\omega^\chi, \omega_c^{\chi_2+i\chi_3})$.

□

Note that $\left(\Omega^2(\tilde{\mathcal{Q}}/\mathfrak{h}_V^\circ)\right)^3$ is a locally constant sheaf on $\mathfrak{h}_V^\circ$ whose stalk at ξ is $\Omega^2(S_\xi, \mathbb{R})$, and $\underline{\omega}_{\mathrm{unf}}$ is a locally flat section of this sheaf.

We end this section by fixing some notation for the family $\tilde{\tilde{\Xi}} : \tilde{\mathcal{Q}} \to \mathfrak{h}_V$. The trivial flat connection will be denoted by $\mathbf{H}_{\mathrm{unf}}$. We also define $\mathbb{C}^*$-actions κ_f on $\tilde{\mathcal{Q}}$ and κ_b as follows: first recall there are $\mathbb{C}^*$-actions ρ_f on $\mathcal{K}$ and ρ_b on $\mathfrak{h}_c$ making Θ equivariant. We define

$$\kappa_f(\lambda)\big(\chi, \chi_c, x\big) := \big(|\lambda|^2\chi, \rho_f(\chi_c, x)\big)$$

$$\kappa_b(\lambda)\big(\chi, \chi_c\big) := \big(|\lambda|^2\chi, \rho_b(\chi_c)\big) \quad \forall \lambda \in \mathbb{C}^*$$

Then clearly equivariance of Θ implies equivariance of $\widetilde{\widetilde{\Xi}}$. Notice also that the definition ensures there is no proper $\mathbb{C}^*$-invariant neighborhood of 0 in $\widetilde{\mathcal{Q}}$. It follows that $\underline{\omega}_{\text{unf}}$ is unique up to scale and that the period map of $\widetilde{\widetilde{\Xi}}$ is $p_W \circ p_I : \mathfrak{h}_V \to \mathfrak{h}_c/W$.

3.2.3 Fibering hyperkähler deformations over a platycosm

Let (Q, δ) be an oriented platycosm, and fix all data as in definition 3.1.11. Our resulting M_0 is then an ADE G_2-platyfold of type Γ, with closed G_2-structure φ_0. We write $(\eta_0, \mu_0, \mathbf{H}_0)$ for its associated Donaldson data; in particular, $\mathbf{H}_0$ is the horizontal distribution associated to the connection ∇ on $\mathbf{V}$, and hence is flat and preserves the vertical hyperkähler structures. Let $\mathfrak{g}$ be the compact Lie algebra associated to Γ, $\mathfrak{h}$ a Cartan subalgebra, and $r := \operatorname{rank}(\mathfrak{g}) = \dim(\mathfrak{h})$.

In this section we will use our adaptation of Kronheimer's construction (specifically Proposition 3.2.7) to build a family of hyperkähler deformations $\mathcal{E}$ parametrized by Q. We will prove that flat sections of this family define 7-manifolds with a closed G_2-structure. Such manifolds appear as subspaces of what is essentially "the Slodowy slice over $\mathcal{E}$". Moreover, the image of such sections can be embedded in T^*Q and admit an interpretation as "flat spectral covers" of Q. To explain what these objects are, let $\mathfrak{h}_Q \to Q$ be the trivial flat bundle of Cartan subalgebras.

Consider the flat bundles:

$$E_W := \mathrm{tot}((\mathfrak{h}_Q \otimes T^*Q)/W) \to Q$$

$$E := \mathrm{tot}(\mathfrak{h}_Q \otimes T^*Q) \to Q \tag{3.2.7}$$

which are $3(r+1)$-dimensional real manifolds. We denote by δ_E the flat structure induced by δ on E. There is a natural projection map $E \to E_W$ which is a $|W|$-to-1 cover with Galois group W. Suppose we have a section $s : Q \hookrightarrow E_W$. We call the restriction $E_W|_{s(Q)} \to Q$ the *spectral cover of* Q *associated to* s. Note that s can be viewed as a multi-section of $T^*Q \to Q$, which is the usual formulation of spectral covers.

The fiber product:

$$\Sigma_s := E \times_{E_W} Q \tag{3.2.8}$$

is called the *cameral cover of* Q *associated to* s. It comes equipped with a natural $|W|$-to-1 map $\Sigma_s \to Q$ and an embedding $\Sigma_s \hookrightarrow E$. Given Σ_s and these two maps, one can recover the section s, and hence the spectral cover.

The spectral covers that will be of interest to us are a slight modification of this example, where we replace E by a flat vector bundle $\mathcal{E}$ with the same fibers, but transition functions "twisted" by $\mathbf{V}$.

Our goal in this section is to prove the following result:

Theorem 3.2.8. *There is a rank $3r$ flat vector bundle $t : \mathcal{E} \to Q$ and a family $u : \mathcal{U} \to \mathcal{E}$ of complex surfaces, equipped with Donaldson data:*

$$\begin{cases} \eta \in \Omega^2(\mathcal{U}/\mathcal{E}) \otimes u^*\Omega^1(\mathcal{E}) \\ \mu \in u^*\Omega^3(\mathcal{E}) \\ \mathbf{H} : u^*T\mathcal{E} \to T\mathcal{U} \ \text{a connection} \end{cases} \tag{3.2.9}$$

The family has the following properties:

1. $\mathcal{U}|_{\underline{\mathbf{0}}(Q)} \cong M_0$

2. $(\eta + \mu)|_{M_0} = \varphi_0$

3. $\mathcal{U}|_{t^{-1}(q)} \cong \mathcal{Q}$

where $\underline{\mathbf{0}} : Q \to \mathcal{E}$ denotes the zero-section.

Moreover, given a flat section $s : Q \to \mathcal{E}$, let $M_s := u^{-1}(s(Q))$. Then the restrictions $(\eta|_{M_s}, \mu|_{M_s}, \mathbf{H}|_{M_s})$ satisfy Donaldson's criteria, and hence define a closed G_2-structure $\varphi_s := (\eta + \mu)|_{M_s}$ on M_s.

Corollary 3.2.9. *Given an ADE G_2-platyfold $(M_0, \varphi_0) \to Q$, there is a* moduli space of closed G_2-deformations *given by:*

$$\mathcal{M}_{G_2}(M_0) := \Gamma_{\mathit{flat}}(Q, \mathcal{E}) \tag{3.2.10}$$

In other words, the deformations of (M_0, φ_0) are parametrized by flat spectral covers of Q.

The main ingredient to prove Theorem 3.2.8 will be, once $\mathcal{E}$ is constructed, to pullback the modified Kronheimer family from last section to $\mathcal{E}$. In order to do that, we need a map $\mathcal{E} \to \mathfrak{h}_V$ *compatible with the flat structure on* $\mathcal{E}$. However, we do not have such a map; indeed, notice that even in cases when $t : \mathcal{E} \to Q$ is trivial as a smooth vector bundle - which happens exactly when $(\mathbf{V}, \nabla)$ is trivial -, it is not trivial as a *flat* vector bundle: the metric connection δ has non-trivial monodromy H_π, and the same is true for the induced flat structure $\delta_{\mathcal{E}}$. To circumvent this issue, we work over a flat trivialization of $\mathcal{E}$, where such maps are available locally; then we glue the pullback families together using the cocycle of $\mathbf{V}$.

Another equivalent formulation would be to work with the pullback of $\mathcal{E}$ to the universal cover $\widetilde{Q} \to Q$. In fact, we can do something simpler: we can work over the *monodromy cover* of Q, i.e., the minimal cover where the monodromy action is trivial. Due to Bieberbach's theorem, the monodromy cover of a platycosm is always a three-torus $\mathbf{T}$. It is defined by a finite unramified covering map $c : \mathbf{T} \to Q$ with Galois group H_π. We then get a *trivial flat bundle* $\widetilde{\mathcal{E}} \to \mathbf{T}$, and we can choose a flat trivialization where $\widetilde{\mathcal{E}} \cong \mathbf{T} \times \mathfrak{h}_V$. This gives us a map:

$$\kappa : \widetilde{\mathcal{E}} \to \mathfrak{h}_V \tag{3.2.11}$$

This is simpler than working in the universal cover because we only need to worry about the action of H_π, i.e. we can forget the lattice $\mathbb{Z}^3 \leqslant \pi$. The drawback of this approach is that one must be careful to choose Donaldson data H_π-invariantly.

We will break the proof of Theorem 3.2.8 into a few lemmas:

Lemma 3.2.10. *There is a vector bundle $t : \mathcal{E} \to Q$ with rank$(\mathcal{E}) = 3r$ and a family $u : \mathcal{U} \to \mathcal{E}$ of complex surfaces satisfying:*

1. $\mathcal{U}|_{\underline{\mathbf{0}}(Q)} \cong M_0$

2. $\mathcal{U}|_{t^{-1}(q)} \cong \mathcal{Q}$

where $\underline{\mathbf{0}} : Q \to \mathcal{E}$ denotes the zero-section.

Proof. Let $\mathfrak{U} := \{U_i; i \in I\}$ be a trivializing flat cover of Q (i.e, $\delta|_{U_i}$ has trivial monodromy) which also trivializes $(\mathbf{V}, \nabla) \to Q$. The argument essentially consists of gluing together "locally constant" copies of $\mathcal{Q} \to \mathfrak{h}_V$ over U_i using the cocycle defining the vector bundle $\mathbf{V}$. The proof in the holomorphic setup is due to Szendrői [Sze04], and it follows through also in our flat setup. We reprodruce it here for completeness.

Let $\zeta_{\mathbf{V}} \in H^1(\mathfrak{U}, C_\Gamma)$ be the cocycle of transition functions of $\mathbf{V} \to Q$. It is valued in C_Γ since $\mathbf{V}$ comes equipped with a compatible fiberwise Γ-action. Since $C_\Gamma \leqslant Sp(1)$, it acts on $\mathfrak{h}_V$ by rotating the hyperkähler classes, so we have a map $C_\Gamma \to GL(\mathfrak{h}_V)$. This induces:

$$\epsilon : H^1(\mathfrak{U}, C_\Gamma) \to H^1(\mathfrak{U}, GL(\mathfrak{h}_V)) \tag{3.2.12}$$

and since $H^1(\mathfrak{U}, GL(\mathfrak{h}_V)) \cong H^1(Q, GL(\mathfrak{h}_V))$, the image $\epsilon(\zeta_{\mathbf{V}})$ defines a rank r vector bundle $t : \mathcal{E} \to Q$. This bundle is trivialized by $\mathfrak{U}$, so we write $\mathcal{E}|_{U_i} \cong U_i \times \mathfrak{h}_V$. This gives us maps

$$\psi_i : \mathcal{E}|_{U_i} \to \mathfrak{h}_V \tag{3.2.13}$$

Note that the metric on Q gives an isomorphism $T^*Q \cong \Lambda^2 TQ$, and this last space is exactly the adjoint bundle of $SU(2)$. This allows us to write $\mathcal{E}|_{U_i} \cong U_i \times \mathfrak{h} \otimes T^*U_i$. The global identification between the flat bundles $\mathcal{E}$ and $\mathfrak{h} \otimes T^*Q$ is given by the inversion condition: let ζ_{T^*Q} and $\zeta_{\mathcal{E}} = \epsilon(\zeta_{\mathbf{V}})$ be the cocycles of the respective bundles. Let $Ad|_{H_\pi} : H_\pi \to SO(V)$ be the standard representation restricted to $H_\pi \subset SO(3)$. This gives $Ad_{H_\pi}(\zeta_{T^*Q}) \in H^1(\mathfrak{U}, SO(V)) \subset H^1(\mathfrak{U}, GL(\mathfrak{h}_V))$ via $A \in SO(3) \mapsto 1 \otimes A \in GL(\mathfrak{h} \otimes V)$. Then $\zeta_{\mathcal{E}} = Ad|_{H_\pi}(\zeta_{T^*Q})^{-1}$.

Now consider the hyperkähler family $\Xi : \mathcal{Q} \to \mathfrak{h}_V$, which is itself a pullback of the Kronheimer miniversal deformation $\Theta : \mathcal{K} \to \mathfrak{h}_c$ by a map $p_2 : \mathfrak{h}_V \to \mathfrak{h}_c$. Let $\mathcal{U}_i := (\psi_i \circ p_2)^* \mathcal{K}$. This gives, for every $i \in I$, a family of complex surfaces:

$$u_i : \mathcal{U}_i \to U_i \times \mathfrak{h}_V \tag{3.2.14}$$

We now glue these families together over the U_i's using the Čech cocycle $\zeta_{\mathbf{V}} \in H^1(\mathfrak{U}, C_\Gamma)$. For this to make sense, we need to realize C_Γ as a subgroup of $Aut(\mathcal{K})$. But this follows from $C_\Gamma \leqslant Aut(\mathbb{C}^2/\Gamma)$ and the fact that this last group acts on $\mathcal{K}$, as $\mathcal{K} \to \mathfrak{h}_c$ is miniversal. So we can think of $\zeta_{\mathbf{V}}$ as an element of $H^1(\mathfrak{U}, Aut(\mathcal{K}))$. Thus, the datum $\{(\mathfrak{U}, \zeta_{\mathbf{V}}, u_i); i \in I\}$ provides us with a family of complex surfaces:

$$u : \mathcal{U} \to \mathcal{E} \tag{3.2.15}$$

and, by construction, $\mathcal{U}|_{t^{-1}(q)} \cong \mathcal{Q}$.

Now restrict u to the zero-section $\underline{\mathbf{0}} : Q \to \mathcal{E}$. Being the zero-section means that $\mathcal{U}|_{\underline{\mathbf{0}}(Q)}$ is glued by local pieces $\Psi^{-1}(0) \times U_i \cong \mathbb{C}^2/\Gamma \times U_i$ according to $\zeta_{\mathbf{V}}$, i.e., $\mathcal{U}|_{\underline{\mathbf{0}}(Q)} \cong M_0$. $\square$

We now have a diagram:

$$\begin{tikzcd} \mathcal{U} \arrow[dd, "q"] \arrow[dr, "u"] & \\ & \mathcal{E} \arrow[dl, "t"] \\ Q & \end{tikzcd} \tag{3.2.16}$$

Lemma 3.2.11. *There are elements:*

$$\begin{cases} \eta \in \Omega^2(\mathcal{U}/\mathcal{E}) \otimes u^*\Omega^1(\mathcal{E}) \\ \mu \in u^*\Omega^3(\mathcal{E}) \end{cases} \tag{3.2.17}$$

satisfying:

$$(\eta + \mu)|_{M_0} = \varphi_0 \tag{3.2.18}$$

Proof. For each trivializing open set U_i, fix an oriented basis of flat local sections $\{\sigma^\circ_{1,i}, \sigma^\circ_{2,i}, \sigma^\circ_{3,i}\} \subset \Omega^1(U_i)$ such that $\eta_0|_{U_i} = \underline{\omega}_{\mathrm{unf}}(0,0,0) \otimes \underline{\sigma}^\circ_i$. On intersections $U_i \cap U_j$ these sections glue according to the monodromy $H_\pi \leqslant SO(3)$ of (Q, δ).

Let $\zeta_{T^*Q} \in H^1(\mathfrak{U}, SO(3))$ be the associated Čech cocycle. Due to the inversion condition, $\underline{\omega}_{\text{unf}}(0,0,0)$ is glued over open sets according to $\zeta_{T^*Q}^{-1}$. Moreover, since (Q,δ) preserves an orientation, $det(\zeta_{T^*Q}) = 1$ and $\sigma^\circ_{1,i} \wedge \sigma^\circ_{2,i} \wedge \sigma^\circ_{3,i}$ is a well-defined global flat 3-form on Q, which we normalize to be equal to the given μ_0. This form pulls-back to a global flat element $t^*\mu_0 \in \Omega^3(\mathcal{E})$. Define:

$$\mu := (u^{-1}t^*)\mu_0 \in u^{-1}\Omega^3(\mathcal{E}) \tag{3.2.19}$$

This is a global section of the sheaf $u^{-1}\Omega^3(\mathcal{E})$ on $\mathcal{U}$. The notation u^{-1} means that we take the subsheaf of the pullback whose sections are constant in the vertical direction.

Now, for $a \in \{1,2,3\}$, consider the pullbacks:

$$\begin{aligned}(\omega_a)_i &:= \psi_i^*(\omega_a)_{\text{unf}} \in \Omega^2(\mathcal{U}_i/\mathcal{E}_i)\\ (\sigma_a)_i &:= u_i^* t_i^* \sigma^\circ_{a,i} \in u_i^*\Omega^1(\mathcal{E}_i)\end{aligned} \tag{3.2.20}$$

where $t_i : \mathcal{E}_i \to U_i$ is the obvious map.

Since the $(\sigma_a)_i$'s are pullbacks of the $\sigma^\circ_{a,i}$, they glue together over $\mathcal{U}$ according to ζ_{T^*Q}. Since $\Omega^2(\mathcal{U}_i/\mathcal{E}_i)$ is glued over the U_{ij}'s according to the Čech cocycle $\epsilon(\zeta_{\mathbf{V}}) = Ad|_{H_\pi}(\zeta_{T^*Q})^{-1}$, it follows that the element:

$$\eta_i := \sum_{a=1}^{3} (\omega_a)_i \otimes (\sigma_a)_i \tag{3.2.21}$$

is such that $\eta_i|_{U_{ij}} = \eta_j|_{U_{ij}}$ so defines a global section:

$$\eta \in \Omega^2(\mathcal{U}/\mathcal{E}) \otimes u^*\Omega^1(\mathcal{E}) \tag{3.2.22}$$

Now consider $(\eta+\mu)|_{M_0}$. For every i, $(\psi_i|_{M_0})^*\underline{\omega}_{\text{unf}}$ is just the hyperkähler structure on the central fiber of the Slodowy slice, glued over Q according to $\zeta_{\mathbf{V}}$. Thus it is clear that $\eta|_{M_0} = \eta_0$ and $\mu|_{M_0} = \mu_0$.

$\square$

Lemma 3.2.12. *Let s be a flat section of $t : \mathcal{E} \to Q$ and let $M_s := u^{-1}(s(Q))$. Define $\pi_s := u|_{M_s} : M_s \to s(Q)$. There is a connection $\mathbf{H}$ on $u : \mathcal{U} \to \mathcal{E}$ such that $\mathbf{H}$ restricts to a flat connection $\mathbf{H}_s$ on π_s.*

Proof. What we want is to show there is a splitting:

$$\begin{tikzcd} 0 \arrow[r] & T(\mathcal{U}/\mathcal{E}) \arrow[r] & T(\mathcal{U}) \arrow[r] & u^*T(\mathcal{E}) \arrow[r] \arrow[l, dotted, bend right, "\exists \mathbf{H}?"'] & 0 \end{tikzcd} \tag{3.2.23}$$

inducing a second splitting:

$$\begin{tikzcd} 0 \arrow[r] & T(M_s/s(Q)) \arrow[r] & T(M_s) \arrow[r] & u^*T(s(Q)) \arrow[r] \arrow[l, dotted, bend right, "\exists \mathbf{H}_s?"'] & 0 \end{tikzcd} \tag{3.2.24}$$

and normalized to restrict at the zero section to:

$$\begin{tikzcd} 0 \arrow[r] & T(M_0/Q) \arrow[r] & T(M_0) \arrow[r] & u^*T(Q) \arrow[r] \arrow[l, bend right, "\mathbf{H}_0"'] & 0 \end{tikzcd} \tag{3.2.25}$$

To construct $\mathbf{H}$, we claim that all we need to do is to define a *partial connection* $\mathbf{H}_q$ on t, i.e., a splitting:

$$
\begin{tikzcd}
0 \arrow[r] & T(\mathcal{U}/\mathcal{E}) \arrow[r] & T(\mathcal{U}/Q) \arrow[r] & u^*T(\mathcal{E}/Q) \arrow[r] \arrow[l, bend right, dotted, "\exists \mathbf{H}_q?"'] & 0
\end{tikzcd}
\tag{3.2.26}
$$

To see why this is so, assume we have constructed $\mathbf{H}_q$ and consider the diagram:

$$
\begin{tikzcd}
0 \arrow[r] & T(\mathcal{U}/\mathcal{E}) \arrow[r, "g'"] \arrow[d, "\|"] & T(\mathcal{U}/Q) \arrow[r, "g"] \arrow[d, "\iota"] & u^*T(\mathcal{E}/Q) \arrow[r] \arrow[d] \arrow[l, bend right, "\mathbf{H}_q"'] & 0 \\
0 \arrow[r] & T(\mathcal{U}/\mathcal{E}) \arrow[r, "f'"] & T(\mathcal{U}) \arrow[r, "f"] \arrow[d] & u^*T(\mathcal{E}) \arrow[r] \arrow[d] \arrow[l, bend right, dotted, "\mathbf{H}"'] \arrow[u, bend right, dotted, "\overline{\delta}_{\mathcal{E}}"'] & 0 \\
 & & q^*T(Q) \arrow[r, "="] & q^*T(Q) \arrow[r] \arrow[u, bend right, "\delta_{\mathcal{E}}"'] & 0
\end{tikzcd}
\tag{3.2.27}
$$

Here $\delta_{\mathcal{E}}$ is the flat connection on $\mathcal{E} \to Q$. We want to define a section $\mathbf{H}$ of f. Because $\delta_{\mathcal{E}}$ splits the last vertical sequence, the section $\overline{\delta}_{\mathcal{E}}$ exists so we can define $\mathbf{H}$ as the composition $\iota \circ \mathbf{H}_q \circ \overline{\delta}_{\mathcal{E}}$.

Now, notice that over each $U_i \times \mathfrak{h}_V$, we can find a copy of $T\mathfrak{h}_V \subset T\mathcal{U}_i$. This is of course just the trivial connection $\psi_i^*\mathbf{H}_{\mathrm{unf}}$ on $\mathcal{U}_i \to \mathfrak{h}_V$. We can glue these together over the U_i's using the cocycle of $\mathcal{U}$ to obtain a distribution $\mathbf{H}_q \subset T\mathcal{U}$. Note that by construction, this distribution is vertical with respect to Q, that is, $\mathbf{H}_q \subset T(\mathcal{U}/Q)$. So it defines a flat partial connection as desired.

Thus, we have defined a connection $\mathbf{H}$ on u. Now we show it restricts to a flat connection $\mathbf{H}_s$ on π_s. In fact, $\mathbf{H}_s$ will only depend on the vertical part $\mathbf{H}_q$ of $\mathbf{H}$:

$$
u^*T(s(Q)) \subset u^*\mathbf{H}_{\delta_{\mathcal{E}}} = u^*t^*TQ = q^*TQ \hookrightarrow T\mathcal{U} \tag{3.2.28}
$$

Here the first containment is the flatness of s, while the last map is the connection $\mathbf{H}_q$. Since we are only looking over $s(Q) \subset \mathcal{E}$, this actually produces a map into $T\mathcal{U}|_{M_s}$. This gives the connection $\mathbf{H}_s$. It is flat because both $\delta_{\mathcal{E}}$ and $\mathbf{H}_q$ are. Moreover, it is clear from the construction of $\mathbf{H}_q$ that $\mathbf{H}|_{\underline{\mathbf{0}}} = \mathbf{H}_0$. □

Lemma 3.2.13. *The restrictions* $(\eta_s, \mu_s, \mathbf{H}_s) := (\eta|_{M_s}, \mu|_{M_s}, \mathbf{H}|_{M_s})$ *satisfy Donaldson's criteria:*

$$\begin{cases} d_f \eta_s = 0 \\ d_{\mathbf{H}_s} \eta_s = 0 \end{cases} \qquad \begin{cases} d_f \mu_s = 0 \\ d_{\mathbf{H}_s} \mu_s = 0 \end{cases} \tag{3.2.29}$$

and hence define a closed G_2-structure $\varphi_s := \eta_s + \mu_s$ on M_s such that $\pi_s : M_s \to Q$ is a coassociative fibration.

Proof. We have proved there is a natural connection $\mathbf{H}$ on $q : \mathcal{U} \to Q$. This allows us to define a *horizontal differential*:

$$d_{\mathbf{H}} : \Omega^{k,l}(\mathcal{U}) \to \Omega^{k,l+1}(\mathcal{U}) \tag{3.2.30}$$

as follows: for horizontal elements, just apply $q^* d_Q$. Consider a vertical element $v \in \Omega^{k,0}(\mathcal{U})$. It represents an element $v' \in \Omega^k(\mathcal{U})$. Then apply $d_{\mathcal{U}} v'$ and project it down to get to $\Omega^{k+1,0}(\mathcal{U})$. Define the result to be $d_H v$. Then one proves that if a different representative $v'' \in \Omega^k(\mathcal{U})$ is chosen, $d_{\mathcal{U}} v' - d_{\mathcal{U}} v''$ is a horizontal form, hence is killed by the projection to $\Omega^{k+1,0}(\mathcal{U})$.

Thus $\mathbf{H}$ induces a decomposition, $d_{\mathcal{U}} = d_f + d_{\mathbf{H}} + F_{\mathbf{H}}$. Since $\mathbf{H}$ is flat, we have

$F_{\mathbf{H}} = 0$.

It is enough to show that Donaldson's criteria are met by the triple $(\eta, \mu, \mathbf{H})$, and the result will follow by restriction to s.

The equations $d_{\mathbf{H}}\mu = 0 = d_f\nu$ follow from the definitions of ν and μ. The equations $d_f\mu = 0$ and $d_{\mathbf{H}}\eta = 0$ then follows from the fact that μ and η are closed.

□

We can now provide a good visualization of our family of 7-manifolds. Consider the diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[d, "w:=\tau^* u"] \arrow[dd, bend right=60, "f"'] & \mathcal{U} \arrow[d, "u"'] \arrow[dd, bend left, "q"] & \mathcal{Q} \arrow[d, "\Xi"'] \\
Q \times H^0_{\mathrm{flat}}(Q, \mathcal{E}) \arrow[r, "\tau"] \arrow[d, "\pi_2"'] & \mathcal{E} \arrow[r, "p_2"] \arrow[d, "t"'] & \mathfrak{h}_V \\
H^0_{\mathrm{flat}}(Q, \mathcal{E}) & Q &
\end{tikzcd} \tag{3.2.31}$$

Here, τ is the *tautological map*: $\tau(q, s) := s(q)$ and $\mathcal{F}$ is the pullback of $\mathcal{U}$ by τ. From now on, we write $\mathcal{B} := H^0_{\mathrm{flat}}(Q, \mathcal{E})$.

Our family of interest is $f : \mathcal{F} \to \mathcal{B}$. For every section $s \in \mathcal{B}$, $M_s = f^{-1}(s)$ is a 7-manifold given by an ALE-fibration over Q, with the fibration given by $\pi_s := w|_{M_s} : M_s \to Q$. Notice that due to the nature of the map τ, different flat sections pick different profiles of ALE-fibers. In particular, $f^{-1}(\mathbf{0}) = M_0$.

One should think of (M_s, φ_s) as a "flat hyperkähler deformation" of (M, φ_0). This picture also provides us with an explicit model for the moduli space of such G_2-structures: it is just the base $\mathcal{B}$, and it only depends on $\mathfrak{h}$ and the flat structure

δ on Q. We will study an explicit model in the next chapters, where $\mathfrak{g} = \mathfrak{su}(n)$ and Q is the Hantzsche-Wendt platycosm. Moreover, we will show that the sections s have an interpretation as an analogue in flat geometry of spectral covers of Higgs bundles.

Remark 3.2.14. Before we end this section, we would like to explain the connection of this construction with a certain "partial topological twist", which was introduced in [Ach98] and discussed in detail in [BCHSN18]. The relevant condition here is that the connection δ is *metric*. The metric on Q induces an isomorphism $T^*Q \cong \Lambda^2 TQ$, and this last space is the bundle of adjoint representations of $SU(2)$. The condition identifies these two as flat bundles, so that locally flat 1-forms can be naturally identified with locally flat adjoint sections. The partial topological twist can then be described as follows: if one starts with a G_2-manifold M fibered by ALE spaces over Q, and global relative 2-cycles $\alpha_i \in H_2(M/Q, \mathbb{R})$, then pairing the G_2-structure φ with the α_i's gives n 1-forms $\theta_i \in \Omega^1(Q)$, each of which has 3 components $\theta_i = (\theta_i^1, \theta_i^2, \theta_i^3)$. On the other hand, at each ALE-fiber, we can associate to $(\alpha_i)_q$ the three periods of the hyperkähler structure. This gives local functions $f_U^i : U \subset Q \to \mathbb{R}^3$. Because $\mathbf{V} \to Q$ is flat, these functions can be glued together to form global functions, up to monodromy of the flat connection. The topological twist requires that the θ_i's agree locally with the f^i's.

What we have argued here goes in the reverse direction: if one starts with the condition, then the deformations of M_0 given by flat sections admit natural closed

G_2-structures.

It would be interesting to give a more "invariant" description of this condition. One idea would be to formulate it in Costello's framework [Cos13]. We believe a more geometric formulation would involve replacing the oriented affine structure on Q by a "$\mathbb{Z}_2$-twisted oriented affine structure", i.e., a $(\mathbb{R}^3, G)$-structure on Q where G fits into the exact sequence $1 \to \mathbb{R}^3 \to G \to SU(2) \to 1$ (the $\mathbb{Z}_2$ here refers to the covering group of $SU(2) \to SO(3)$). This would possibly extend the construction to a class of spaces slightly more general than the platycosms.

3.3 The Hantzsche-Wendt G_2-platyfold

In section 3.1 we explained how to construct closed G_2-structures on ADE G_2-platyfolds. The construction essentially relies on matching two monodromy actions, one coming from the flat base, and another coming from a flat bundle. In particular, if one starts with a trivial flat bundle, there are no obstructions. In view of the results of the previous section, it is natural to ask what is the deformation space for these G_2-structures. The construction of the deformation family simplifies considerably when $(\mathbf{V}, \nabla)$ is trivial, so we will focus on this case.

In the next chapter, we will give a geometric interpretation for the flat sections parametrizing deformations of G_2-structures in terms of certain "flat Higgs bundles". The interpretation comes from M-theory/IIA duality, which requires us to study M-theory compactified on our G_2-manifold. However, not all ADE G_2-

platyfolds define $\mathcal{N} = 1$ compactifications (as required by M-theory). Among the examples we have discussed, 3.1.6 is the sole one that passes this test; hence, we will use it as our main testing ground in studying the predictions of string dualities.

Definition 3.3.1. The ADE G_2-platyfold $M_0 := \mathbb{C}^2/\Gamma \times_{\mathbb{K}} \mathcal{G}_6$ will be called the *Hantzsche-Wendt G_2-platyfold of type* Γ.

The reason why only M_0 admits a sensible compactification is the following: M-theory compactified on a G_2-space M is a $\mathcal{N} = 1$ supersymmetric theory. Mathematically, this means that the space of parallel spinor fields on M is one-dimensional. If M is an ADE G_2-platyfold over Q, then the holonomy of its Riemannian metric is M is $H = SU(2) \rtimes H_Q \subset G_2$. By the Berger-Wang classification of Riemannian holonomies, the $\mathcal{N} = 1$ condition holds if and only if H_Q is not conjugate to a subgroup of $SU(3) \subset G_2$.

The holonomies H_Q of the orientable platycosms are classified by Theorem 2.1.3, and they are all finite subgroups of $SO(3)$. From the classification of finite subgroups of $SO(3)$, we see that the finite cyclic subgroups necessarily fix an axis in $\mathbb{R}^3$; this automatically implies that in such cases, the holonomy can be conjugated to $SU(3)$. The only remaining group $H_{\mathcal{G}_6} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathbb{R}^3$ without fixing an axis (see equation 2.1.3), and it cannot be conjugated to a subgroup of $SU(3)$.

Of course, one could go around this issue by making $(\mathbf{V}, \nabla)$ nontrivial. In fact, as the examples in section 3.1 suggests, it is possible that the monodromy of ∇ can supply extra factors to the Riemannian holonomy group, and hence generate sensible M-theory compactifications based on the other platycosms. E.g., one could take a flat bundle over $\mathbf{T}$ with monodromy one of the non-cyclic finite subgroups of $SO(3)$; these are D_{2n}, A_4, S_4 and A_5. Another interesting question that we will not touch here is determining whether a deformation of an ADE G_2-platyfold can acquire a metric with full holonomy G_2.

3.4 Associative deformations of the zero-section

In this section we prove that the flat three-torus $Q = \mathbf{T}^3$ is the only closed orientable 3-manifold that is non-rigid as an associative zero-section of an ADE G_2-orbifold $M \to Q$. As Proposition 3.4.1 below shows, the spinor bundle $\$_Q \to Q$ is a trivial (smooth) bundle. Moreover, since $Spin(3) = SU(2)$, $\$_Q$ is the bundle of adjoint representations of $SU(2)$, i.e., it is exactly the (smooth) vector bundle $\mathbf{V}$. In particular, a choice of flat quaternionic connection ∇ and McKay group Γ induce an ADE G_2-orbifold $M \to Q$ by $M \cong \$_Q/\Gamma$. We will prove that the result for $\mathbf{V} \to Q$, and hence the same will hold for M. In particular, it will follow that $\mathcal{G}_6$ is rigid as an associative submanifold of M_0.

Remark: Every closed orientable 3-manifold Q is spin. This is because one can prove there is a covering map $Q \to \mathbf{S}^3$ branched in the complement of an open ball $D^3 \subseteq \mathbf{S}^3$. But $T^*\mathbf{S}^3$ is trivial, and the obstruction to extend a trivialization over D^3 lies in $\pi_2 SO(3) = 0$.

The zero section $Z(\$_Q) \cong Q$ is an example of an *associative submanifold*: this means that $\varphi|_Q = \mathrm{vol}_Q$ (equivalently, $\star\varphi|_Q = 0$). The deformation theory of associative submanifolds of a G_2-manifold is generally obstructed. Deformations of Q as an associative submanifold of $\$_Q$ are given by sections of the normal bundle $\mathcal{N}_{Q/\$_Q}$ preserving the associativity condition. This normal bundle can be identified (see McLean) with a bundle of twisted spinors, $\$(Q) \otimes_{\mathbb{H}} E$. Then the condition on the section is that it is a *harmonic twisted spinor*. However, because we are working on the total space of $\$_Q \to Q$ itself, there is no twist (E is trivial), so the associativity condition on the section is just that it is a harmonic spinor. This is exactly what happens in this case:

Proposition 3.4.1. *Let Q be an orientable 3-manifold. Then $\$_Q$ is a trivial bundle.*

Proof. $\$_Q$ is the vector bundle associated to the principal spin bundle $Spin(Q) \to Q$ via the spinor representation. Let $f : Q \to BSpin(3)$ be the classifying map. Then $\$_Q$ is trivial $\iff$ f is null-homotopic. But $BSpin(3)$ is the 3-connected cover[7] of $BSO(3)$, that is, $\pi_3(BSpin(3)) = 0$. Since $\dim Q = 3$, the result follows. $\square$

Remark 3.4.2. Orientable 3-manifolds are parallelizable, i.e., any map $Q \to BSO(n)$ is null-homotopic.

Finally, we notice that if Q is an orientable platycosm (hence the scalar curvature vanishes), the Lichnerowicz-Weitzenböck formula for the Dirac operator shows that

[7]This is a nontrivial fact, and it actually holds for $n \geqslant 3$. For lower n, $BSpin(n)$ is only the 2-connected cover.

harmonic spinors are parallel. Thus, if e.g. $Q = \mathbf{T}$ is a flat 3-torus, any spin structure has parallel spinors, and thus $\mathbf{T}$ is deformable as an associative in $M = \$(T^3)$. In fact, this is the only non-rigid example:

Theorem 3.4.3. *Let Q be a compact, connected three-manifold, and assume $\$_Q$ admits a G_2-structure. Then the associative $Q \subset \$_Q$ is non-rigid $\iff$ Q is the flat three-torus.*

Proof. As mentioned before, a deformation of Q is given by a parallel spinor $\psi \in H^0_{\mathrm{flat}}(Q, \$_Q)$. If $\psi \neq 0$, then Q is known to be Ricci-flat, hence flat since $\dim Q = 3$. This implies that i) Q is a Bieberbach manifold and ii) ψ is harmonic. But Pfäffle showed ([Pfä00] Theorem 5.1) that the only flat spin manifold where the Dirac operator has non-trivial kernel is the three-torus with its trivial spin structure. □

This applies in particular to $Q = \mathcal{G}_6$. This manifold inherits half of the spin structures of T^3 under the pushforward by $T^3 \to \mathcal{G}_6$. Regardless of which one is chosen, Proposition 3.4.1 implies that $\mathcal{G}_6 \times \mathbb{C}^2 \cong \$_{\mathcal{G}_6}$ and hence by Theorem 3.4.3 the zero-section is rigid as an associative submanifold. We have proved:

Proposition 3.4.4. *$\mathcal{G}_6$ is rigid as an associative submanifold of M_0.*

3.5 Coassociative deformations of the fibers

Coassociative submanifolds are four-dimensional submanifolds S of a G_2-manifold (M, φ) such that $\varphi|_S = 0$ (equivalently, $\star\varphi|_S = \mathrm{vol}_S$). Coassociative deformations are described by the following result of McLean:

Theorem 3.5.1. *(McLean): Deformations of compact coassociatives submanifolds S of a G_2-manifold M are unobstructed, and the moduli space of coassociative deformations $\mathcal{C}_S$ can be identified with an open set of $H^2_+(S,\mathbb{R})$. Hence $\mathcal{C}_S$ is a smooth manifold of dimension $b^2_+(S)$.*

Suppose $M \to Q$ is a coassociative fibration by ALE-spaces S. Since S is hyperkähler, $H^2_+(S_q,\mathbb{R}) \cong \mathbb{R}^3$ is identified with the adjoint representation. The metric on Q gives $T^*Q \cong \Lambda^2 TQ$, which is also the adjoint representation. Hence $T_{[S_q]}\mathcal{C} \cong H^2_+(S_q,\mathbb{R}) \cong T^*Q$ and hence Donaldson data can be also understood as describing simultaneous coassociative deformations of all fibers.

Chapter 4

Type IIA duals

4.1 M-theory/IIA duality

M-theory/IIA duality is a well-known symmetry of string/M-theory generalizing Kaluza and Klein's electrogravity unification. In our context, one can think of it as a map between geometric structures on a G_2-space M and a dual Calabi-Yau manifold X. Additionally, in the presence of $M2/D6$-branes, the duality relates a locus of ADE singularities of type Γ on M to a G-connection on a dual special Lagrangian submanifold of X [Sen97] [AW01]. Here, G is the compact real Lie group that is McKay dual to the finite ADE group Γ.

Physically, one imagines a two-cycle on the desingularization of M (a "$M2$-brane") being blown-down to a singularity of type Γ, and hence becoming massless[1]. The dual description is given by a configuration of $r = \mathrm{rank}(G)$ "$D6$-branes", i.e.,

[1] The mass of a membrane is proportional to its area.

complex line bundles with $U(1)$-connections on special Lagrangian submanifolds of X, with open strings stretching between them. In the massless limit, the $D6$-branes are smashed together on a single submanifold, with a distribution pattern dictated by the Dynkin graph of Γ, and thus producing a rank r vector bundle with a G-connection.

In this chapter we study the moduli spaces of IIA duals to singular M-theory compactifications on ADE G_2-orbifolds $M \to Q$. The *M-theory moduli space*[2] $\mathcal{M}_{G_2}^{\mathbb{C}}$ is a "complexification" of the moduli space $\mathcal{M}_{G_2}$ we studied in the first chapter. The IIA picture is useful because, as we will show, the relevant moduli space parametrizes certain "flat Higgs bundles" (E, θ) on Q. Both E and θ depend non-trivially on the G_2-structure φ and its complexification data. In the next chapter, we prove a spectral correspondence for (E, θ) that allows us to "untwist" the Higgs data and identifies $\mathcal{M}_{G_2} \subset \mathcal{M}_{G_2}^{\mathbb{C}}$ with the moduli of "flat spectral covers", i.e., the base of the Hitchin system for (E, θ). This gives an algebro-geometric interpretation to the G_2-deformations described in the previous chapter.

Remark 4.1.1. To be more precise, we will be studying the gauge-theoretic descriptions of M-theory and IIA strings, which are sometimes refered in the physics literature as the weakly-coupled limits. In this limit, the theories are described as supersymmetric gauge theories in 11 and 10 dimensions, respectively. The type of

[2]For the physicist reader, we note that we consider M-theory with a vanishing cosmological constant. If this requirement is dropped, the moduli space is enhanced to include the moduli of all G-connections on M, where G is the gauge group.

geometric structure arising in the compactification manifolds M and X are determined by dimensionally reducing the equation for a 11 or 10 dimensional parallel spinor down to 7 or 6 dimensions. In the first case, one obtains stationary points of the 7-dimensional Chern-Simons functional $CS(\varphi_{\mathbb{C}}) := \int_M \varphi_{\mathbb{C}} \wedge d\varphi_{\mathbb{C}}$, which are exactly the integrable complexified G_2-structures. In the second case, one obtains the Hermitian-Yang-Mills equations.

Suppose $M \to Q$ is an ADE G_2-orbifold (of type A_n for concreteness), endowed with a $U(1)$-action by isometries with fixed set $Q \subset M$. In analogy with Kaluza-Klein theory, we would like to define the space of orbits $X := M/U(1)$ as the type IIA dual of M. The orbit map $d : M \to X$ maps Q homeomorphically to $d(Q)$, so by abuse of notation we also denote the latter by Q.[3]

The Calabi-Yau space X is called a *IIA dual* for M if it satisfies the following [Ach00] [AW01]:

1. $X := M/U(1)$ as smooth spaces.
2. The complex structure J on X has a real structure such that Q is a totally real special Lagrangian submanifold.
3. There are n $D6$-branes "wrapping" $Q \subset X$.

We note that the $D6$-branes in this setup fill the noncompact spacetime direction

[3]Note that $d(Q)$ is more precisely seen as a singular stratum of X, but we will not make explicit use of this extra structure in what follows.

$\mathbb{R}^{3,1}$. Hence, being $6 + 1$-dimensional objects, they must be supported on a 3-cycle. The special Lagrangian requirement is part of the supersymmetry condition.

These conditions do not say anything about the Calabi-Yau metric on X, so it is not clear how to construct it. If our X is near the large volume limit, the metric should be semi-flat and thus must be determined from a condition on the Hessian metric on the base $\mathcal{B}$ of the SYZ fibration of X. In the next section, we will provide evidence that the condition should be that $\mathcal{B}$ can be identified with an open set of an orbit in a moduli space of monopoles.

The IIA moduli space $\mathcal{M}_{IIA}$ parametrizes the following objects:

1. Complex structures on X in which Q is totally real.

2. Complexified Kähler structures on X

3. A supersymmetric configuration of n $D6$-branes wrapped on Q.

Remark 4.1.2. For the local model $X = T^*Q$, once the metric on Q is fixed, there is a unique complex structure on T^*Q under which Q is totally real. This is the complex structure that makes the semi-flat metric on T^*Q a Calabi-Yau metric.

4.2 The Acharya-Pantev-Wijnholt system

We now analyze the supersymmetry condition for a configuration of n $D6$-branes wrapping Q. If we work with any Calabi-Yau, the moduli space of such configurations receive corrections from holomorphic disks bounded by the branes. However, if we assume X is near the large volume limit, such corrections are small and we can replace X by the symplectic linearization T^*Q. We will argue then that the brane configuration is described by a triple (E, A_c, h) consisting of a rank n complex vector bundle, a "stable" flat $SL(n, \mathbb{C})$-connection A_c on Q, and a harmonic metric h on E. The equation determining h is a moment map condition that selects a prefered gauge orbit of A_c under complexified gauge transformations. In the next section we will give an interpretation of this data by using Corlette's theorem to introduce a moduli space of "flat Higgs bundles" on Q.

Remark 4.2.1. It is known [Wit96] that a configuration of n $D6$-branes on a fixed special Lagrangian Q is described by a $SU(n)$-connection on Q. In our setup, the extra complexified directions parametrize Kähler deformations that keep Q special Lagrangian.

The unbroken supersymmetry (BPS) condition for IIA string theory on T^*Q with n $D6$-branes is given by the *Hermitian-Yang-Mills equations*:

$$\begin{cases} \mathcal{F}^{2,0} = 0 \\ \Lambda\mathcal{F} = 0 \end{cases} \tag{4.2.1}$$

Here $\mathcal{F}$ is the curvature of a $SU(n)$-connection $\mathcal{A}$ on a holomorphic vector bundle $\mathcal{E}$ over T^*Q endowed with a hermitian metric, and Λ is the Lefschetz operator of contraction by the Kähler form. Note that because $\mathcal{A}$ is hermitian, the first equation implies $\mathcal{F}^{0,2} = \overline{\mathcal{F}^{2,0}} = 0$.

It is known that the supersymmetry condition for *one* $D6$-brane wrapping Q is given by a flat connection on a complex line bundle over Q. This is a dimensional reduction of a $U(1)$ Hermitian-Yang-Mills connection on a line bundle over T^*Q. Thus, to find the condition on Q, we need to compute the dimensional reduction of equations 4.2.1 down to Q.[4]

Since we are working on T^*Q, choose local coordinates x_j on Q and y_j on the fiber, $j = 1, 2, 3$. The complex structure J adapted to the semi-flat metric is chosen such that $z_j = x_j + iy_j$ are holomorphic coordinates (i.e., Q is totally real). Now, the hermitian condition allows us to write $\overline{\mathcal{A}}^{1,0} = \mathcal{A}^{0,1}$, and it follows that:

$$
\begin{aligned}
\mathcal{A}_j^{1,0} &= \mathcal{A}_j + i\mathcal{A}_{j+3} \\
\mathcal{A}_j^{0,1} &= \mathcal{A}_j - i\mathcal{A}_{j+3}
\end{aligned}
\tag{4.2.2}
$$

Let us write $\mathcal{A}^{1,0} = \sum_{j=1}^{3} A_j dx_j + i\theta_j dy_j$. Assume that A and θ do not depend on the fiber directions y_j. It is clear that:

$$
A := \sum_{j=1}^{3} A_j dx_j \tag{4.2.3}
$$

becomes a well-defined $\mathfrak{su}(n)$-valued one-form on Q. Moreover, because Q is special Lagrangian, $\mathcal{N}_{Q/T^*Q} \cong TQ$ so the *a priori* vertical $\mathfrak{su}(n)$-valued one-form

[4] Another way to see this is by noting that the HYM equation describes a “spacetime filling brane” on T^*Q, and the wrapping condition is obtained by performing three T-dualities along the fiber directions.

$$\sum_{j=1}^{3} \theta_j dy_j \tag{4.2.4}$$

can be be dualized (via the metric) to a $\mathfrak{su}(n)$-valued one-form θ. In this semi-flat setup, the dualization map is the action of the complex structure $J_k^j(dy_j) = -dx_k$.

Let $d = \partial + \overline{\partial}$. By assumption, $\overline{\partial}A = \overline{\partial}\theta = 0$. We have:

$$\begin{aligned} \mathcal{F}^{2,0} &= \partial\mathcal{A}^{1,0} + \mathcal{A}^{1,0} \wedge \mathcal{A}^{1,0} \\ &= \partial(A + i\theta) + (A + i\theta) \wedge (A + i\theta) \end{aligned} \tag{4.2.5}$$

and

$$\mathcal{F}^{1,1} = \partial\mathcal{A}^{0,1} + \mathcal{A}^{1,0} \wedge \mathcal{A}^{0,1} \tag{4.2.6}$$

The first equation becomes:

$$\begin{aligned} F_A &= \theta \wedge \theta \\ D_A\theta &= 0 \end{aligned} \tag{4.2.7}$$

and the second equation is:

$$D_A \star \theta = 0 \tag{4.2.8}$$

where the "bundle Hodge star" on $\Omega^1(Ad(\mathcal{E}|_Q))$ is a combination of the Hodge star on the base Q and the Hodge star induced by the hermitian metric on $\mathcal{E}$. One can also define the "adjoint" of D_A by $D_A^\dagger := \star D_A \star$ and write equation 4.2.8 as $D_A^\dagger \theta = 0$.

We will refer to equations 4.2.7 and 4.2.8 as the *Acharya-Pantev-Wijnholt system*, or APW for short. The reason we choose this name is that, as far as the author is aware, the recognition that these equations describe the supersymmetric gauge theory associated to a system of D_6-branes wrapping a three-cycle appears first in work of Acharya [Ach98], and the Higgs bundle/spectral cover interpretation, which we will discuss next, first appeared in the work of Pantev and Wijnholt [PW11]. Recently, the system has been studied more carefully in [BCHSN18] and [BCHLTZ18]. We note, however, that these equations have appeared long before (in a different context) in the Mathematics literature in the works of Donaldson [Don87] and Corlette [Cor88]. In fact, theorem 5.1.8 below establishes that solutions to these equations are essentially described by the well-known Donaldson-Corlette theorem 5.1.7.

4.3 The Hantzsche-Wendt Calabi-Yau

4.3.1 SYZ fibration and special Lagrangian deformations

The celebrated *SYZ Conjecture* [SYZ96] is a geometric formulation of Mirror Symmetry. In its essence, it claims that mirror Calabi-Yau manifolds should admit dual special Lagrangian torus fibrations. Here, "dual" means that a torus fiber

$\mathbf{T}_b$ in a fibration $g : Z \to B$ is the moduli spaces of flat $U(1)$-connections of the corresponding fiber $\widehat{\mathbf{T}}_b$ in the dual fibration $\widehat{Z} \to B$, and vice-versa.

This description, however, is only true generically: in practice, one has to allow fibrations with "special fibers" along a singular locus $\Delta \subset B$. Such fibers can be either singular limits, finite quotients or even "collapsed limits" of the smooth torus fibers. There are local models for the SYZ fibrations over subsets of Δ, and these admit a semi-flat Calabi-Yau metric [LYZ04]. For these metrics, the smooth fibers are always special Lagrangian. The study of mirror symmetry for such Calabi-Yau spaces is often called *semi-flat Mirror Symmetry*. In this approach, one builds the Calabi-Yau structure on the mirror $\hat{g} : \hat{Z} \to B$ by first dualizing the smooth part of $g : Z \to B$, and then correcting the dual semi-flat structure on $\hat{g}$ by "instanton corrections" depending only on the structure of the singular locus Δ of g.

In chapter 3, we proved that the Hantzsche-Wendt G_2-platyfold M admits a sensible M-theory compactification. In the previous section, we showed that the classical approximation to M-theory/IIA duality leads to a description of the G_2-deformations of M in terms of A-branes on its IIA dual, the *Hantzsche-Wendt Calabi-Yau* $T^*\mathcal{G}_6$. In this section we study the SYZ-fibration structure of $X := T^*\mathcal{G}_6$. We will define a torus fibration with total space X and will discuss how to pick the correct semi-flat Calabi-Yau metric.

We first construct the fibration. Recall that we have the usual cotangent map $T^*\mathbf{T} \to \mathbf{T}$. This is in fact a trivial flat bundle - i.e., the metric connection has no monodromy. It follows that there is a canonical way of identifying all fibers: fix a base point $0 \in \mathbf{T}$ and isomorphisms $i_x : T^*_x\mathbf{T} \cong T^*_0\mathbf{T}, \forall x \in \mathbf{T}$. Then there is a "fiber projection map":

$$\begin{aligned} \overline{f} : T^*\mathbf{T} &\to T^*_0\mathbf{T} \\ (x, \xi) &\mapsto i_x(\xi) \end{aligned} \tag{4.3.1}$$

This induces a projection on the $\mathbb{K}$-quotient:

$$T^*\mathbf{T}\big/\mathbb{K} \to T^*_0\mathbf{T}\big/\mathbb{K} \tag{4.3.2}$$

where the $\mathbb{K}$-action on $T^*\mathbf{T}$ is just the usual action on the base $\mathbf{T}$ coupled with the induced action on covectors.

We will need the following lemma:

Lemma 4.3.1. *$X \cong (T^*\boldsymbol{T})/\mathbb{K}$ as smooth manifolds.*

Proof. Recall that $X := T^*\mathcal{G}_6 = T^*(\mathbf{T}/\mathbb{K})$. We can also give a different characterization of X as a $\mathbb{K}$-space: $X = (T^*\mathbf{T})/\widetilde{\mathbb{K}}$, where $\widetilde{\mathbb{K}}$ is another copy of $\mathbb{K}$ acting the usual way on $\mathbf{T}$ and trivially on the cotangent directions. The $\mathbb{K}$ and $\widetilde{\mathbb{K}}$ actions on $T^*\mathbf{T}$ commute so there is an isomorphism:

$$(T^*\mathbf{T})/\mathbb{K}\Big/\widetilde{\mathbb{K}} \cong T^*(\mathbf{T}/\mathbb{K})\Big/\mathbb{K} \tag{4.3.3}$$

Call this space $\mathcal{S}$. So we have two Galois coverings over $\mathcal{S}$ with Galois groups $\mathbb{K}$ and $\widetilde{\mathbb{K}}$:

$$
\begin{tikzcd}
T^*(\mathbf{T}/\mathbb{K}) \arrow[d, "\mathbb{K}"'] & (T^*\mathbf{T})/\mathbb{K} \arrow[d, "\widetilde{\mathbb{K}}"] \\
\mathcal{S} \arrow[r, "\cong"] & \mathcal{S}
\end{tikzcd}
$$

Because the actions involved are free, it is easy to check that $\pi_1(\mathcal{S})$ is an extension of $\mathbb{Z}^3$ by $\mathbb{K} \times \widetilde{\mathbb{K}}$. The fundamental groups of the covering spaces are extensions of $\pi_1(\mathbf{T})$ by actions of $\mathbb{K}$ or $\widetilde{\mathbb{K}}$. However, since the actions are identical on $\mathbf{T}$, the same is true on $\pi_1(\mathbf{T})$, hence they are isomorphic extensions of $\pi_1(\mathbf{T})$. This is, of course, just the group $\pi = \pi_1(\mathcal{G}_6)$. The covering maps send π to two conjugate index four subgroups of $\pi_1(\mathcal{S})$, hence they define isomorphic covering maps, and in particular diffeomorphic total spaces. $\square$

From the lemma, we get our desired fibration:

$$
f : X \to \underbrace{T_0^*\mathbf{T}/\mathbb{K}}_{:=\mathbb{R}^3_{\mathbf{Y}}} \tag{4.3.4}
$$

To check that this can be given a structure of special Lagrangian torus fibration, we first identify $T_0^*\mathbf{T}/\mathbb{K}$ more carefully. Clearly $T_0^*\mathbf{T} \cong \mathbb{R}^3$ and each generator of the $\mathbb{K}$-action just inverts the signs in two of the three directions - i.e., $\mathbb{K} \cong H_{\mathcal{G}_6}$ acts via the presentation 2.1.3. Hence, the base $T_0^*\mathbf{T}/\mathbb{K}$ can be identified with $\mathbb{R}^3$ with a trident-shaped singularity along the three axes; we denote this singularity by $\mathbf{Y}$ for obvious reasons.[5] Each coordinate axis has isotropy $\mathbb{Z}_2$ - coming from the three

[5] Such a trident-shaped space will appear often throughout this work, so we adopt the following convention: $\mathbf{Y}$ is the union the three axes in $\mathbb{R}^3$, and for a topological space A the notation $\mathbf{Y}_A$ means three copies of A joined together at a point. The orbifold structure on these spaces in

order two subgroups of $\mathbb{K}$ - while the origin has isotropy $\mathbb{K}$. We denote this singular space by $\mathbb{R}^3_{\mathbf{Y}}$, and the discriminant locus Δ by $\mathbf{Y}$.

We now discuss the fibers:

- $x \notin \mathbf{Y}$: $f^{-1}(x)$ is a smooth fiber isomorphic to $\mathbf{T}$
- $x \in \mathbf{Y}$, $x \neq 0$: $f^{-1}(x)$ is a reducible scheme[6] $\mathcal{G}_2(\mathbb{Z}_2)$
- $0 \in \mathbf{Y}$: $f^{-1}(0)$ is a reducible scheme $\mathcal{G}_6(\mathbb{K})$

The notation N^A means that the $\mathbb{R}$-scheme $N(A)$ is topologically N and its sheaf of functions is an extension of $\mathcal{C}^\infty_N$ by A. One should think of N as a subscheme of N^A whose normal vectors are parametrized by A.

We now discuss the Calabi-Yau structure on X inducing a special Lagrangian structure on the fibration $X \to \mathbb{R}^3_{\mathbf{Y}}$.

The space $\mathbb{R}^3_{\mathbf{Y}}$ is an example of a so-called $\boldsymbol{Y}$*-vertex* [LYZ04]:

Theorem 4.3.2. *(Loftin, Yau, Zaslow:) $\mathbb{R}^3_{\mathbf{Y}} \backslash \mathbf{Y}$ admits affine Hessian metrics solving the Monge-Ampère equation.*

It follows that $T^*(\mathbb{R}^3_{\mathbf{Y}} \backslash \mathbf{Y})$, and hence also $X \backslash f^{-1}(\mathbf{Y})$, admit semi-flat Calabi-Yau metrics.

We assume the Calabi-Yau structure on X is near the large complex structure limit point on the moduli space. In this limit, the volumes of the torus fibers are

different situations will either be specified separately or will be clear from context.

[6] Recall that $\mathcal{G}_2$ is the unique oriented platycosm with holonomy $\mathbb{Z}_2$, also known as *dicosm*.

small, and the Calabi-Yau metric is approximated by a semi-flat metric on the fibration whose smooth fibers are special Lagrangian tori. Theorem 4.3.2 provides many possible non-trivial semi-flat Calabi-Yau structures on X, and one of them is IIA-dual to the G_2-structure on the Hantzsche-Wendt G_2-platyfold. We will now give a heuristic argument on how the correct metric can be identified.

We first note that the structure of the singular fibers of X is suspiciously similar to the configuration space of $SU(2)$ A-branes on the IIA dual $C^\vee$ of a $\mathbb{Z}_2$-quotient of the G_2-cone $C(SU(3)/U(1)^2)$ [AW01], 3.7 (II). In fact, there is an obvious relation between X and $C^\vee$: $SO(3)$ acts on X by isometries on each SYZ fiber, and on $SU(3)/U(1)^2$ by left multiplication. The orbit space for the first action is $\mathbb{R}^3_{\mathbf{Y}}$; for the second action, it is a certain one point compactification[7] $\mathbb{R}^3_{\mathbf{Y}} \cup \{\infty\}$. This last space can also be described as the Grassmanian $O(3)/O(1)^3$ of triples of oriented lines in $\mathbb{R}^3$; a second, more useful description, is that $\mathbb{R}^3_{\mathbf{Y}} \cup \{\infty\}$ is the orbit of the Atiyah-Hitchin moduli space for two *distinct* $SU(2)$-monopoles of charge 2.

We conjecture that $C^\vee$ is the large volume limit point of X. One idea to prove this is to show that there is a Gromov-Hausdorff collapse of X collapsing the smooth fibers and the most singular fiber $\mathcal{G}_6$ to points, while the other singular fibers $\mathcal{G}_2$ collapse to the flat 2-dimensional orbifold $D^2(2,2;)$ known as the half-pillowcase [BDP17]. The intuition is that when the singular fibers become large, this provides exactly three copies of $\mathbb{R}^3$ touching at a point, each with multiplicity 2. If this is

[7]This is not the usual one-point compactification; the basis around ∞ has to be reduced to not intersect the singular rays.

true, one expects that there is a G_2-deformation connecting their M-theory duals; i.e., that $\widetilde{\mathbb{C}^2/\mathbb{Z}_2} \times_{\mathbb{K}} \mathcal{G}_6$ admits a metric degeneration to $C(SU(3)/U(1)^2)/\mathbb{Z}_2$.

We will come back to this conjecture at the end of this section, after we discuss flat deformations. We will also give more context for it in Chapter 6, where we will prove that the mirror $\widehat{X}$ admits a "smoothing" given by the moduli space of $SO(4,\mathbb{C})$-flat Higgs bundles on the central fiber $\mathcal{G}_6$.[8]

Going back to the fibration $f : X \to \mathbb{R}^3_{\mathbf{Y}}$, the special Lagrangian structure on the fibers can be understood from the proof of McLean's theorem:

Theorem 4.3.3. *(McLean): Deformations of compact special Lagrangian submanifolds S of a Calabi-Yau manifold X are unobstructed, and the moduli space $\mathcal{L}$ is smooth of dimension $b^1(S)$.*

Proof. Each normal field n on S defines by contraction a 1-form $\iota(n)\omega$ and a 2-form $\iota(n)Im(\Theta)$. These are well-defined because $\omega|_S = Im(\Theta)|_S = 0$. Moreover one can prove:

$$\iota(n)\omega = - * \iota(n)Im(\Theta) \tag{4.3.5}$$

The deformation of S associated to n is determined by the exponential map $\exp_n : S \to X$ by $S_n := \exp_n(S)$. S_n is special Lagrangian if and only if $\omega|_{S_n} =$

[8]We expect that, if the conjecture is true, the correct metric to choose on $\mathbb{R}^3_{\mathbf{Y}}$ in order to define the semi-flat Calabi-Yau structure on X should be the decompactification of the homogeneous metric on the generic orbit of the Atiyah-Hitchin moduli space of centered $SU(2)$-monopoles of charge 2. The author is not aware if this is a Hessian metric, though.

$Im(\Theta)|_{S_n} = 0$ if and only if $\exp^* \omega|_S = \exp^* Im(\Theta)|_S = 0$. In other words, the local moduli space $\mathcal{L}_S$ is given by $f_S^{-1}(0)$, where:

$$f_S : U \subseteq \Gamma(\mathcal{N}_{S/X}) \to \Omega^1_S \oplus \Omega^2_S$$

$$n \mapsto (\exp_n^* \omega|_S, \exp^* Im(\Theta)|_S)$$

The tangent space to $\mathcal{L}$ is given by $\ker(df_S(0))$ which after a computation using Lie derivatives reduces to:

$$d\iota(n)\omega = 0 \tag{4.3.6}$$

$$d\iota(n)Im(\Theta) = 0 \tag{4.3.7}$$

Because of 4.3.5, S_n is special Lagrangian if and only if $\iota(n)\omega$ is harmonic. The Hodge theorem now shows that $T_{[S]}\mathcal{L} \cong H^1(S)$. $\square$

It follows that the moduli space of special Lagrangian deformations of the smooth torus fibers of $f : X \to \mathbb{R}^3_{\mathbf{Y}}$ is isomorphic to the complement of the discriminant locus $\mathbb{R}^3_{\mathbf{Y}} \backslash \mathbf{Y}$.

Recall that $\mathcal{G}_6$ embeds in X as the zero-section of $T^*\mathcal{G}_6 \to \mathcal{G}_6$. This copy of $\mathcal{G}_6$ is special Lagrangian with respect to the standard flat metric on $T^*\mathcal{G}_6 \to \mathcal{G}_6$. Assume it is also special Lagrangian with respect to the Calabi-Yau structure on $X \to \mathbb{R}^3_{\mathbf{Y}}$. If this is the case, then:

Corollary 4.3.4. *$\mathcal{G}_6$ is rigid as a special Lagrangian in $X \to \mathbb{R}^3_{\mathbf{Y}}$.*

Proof. The homology of $\mathcal{G}_6$ is $(\mathbb{Z}, \mathbb{Z}_4 \oplus \mathbb{Z}_4, 0, \mathbb{Z})$. Since $\mathcal{G}_6$ is orientable, the result follows from Poincaré duality and Theorem 4.3.3. $\square$

Now we will argue that *flat* deformations of $\mathcal{G}_6 \hookrightarrow X$ are related to special Lagrangian deformations of the fibers of $X \to \mathbb{R}^3_{\mathbf{Y}}$. Consider $\mathcal{G}_6$ as a compact flat manifold. The *Teichmüller space of flat deformations* is given by [BDP17]:

$$\mathbb{T}(\mathcal{G}_6) \cong \left(\mathbb{R}^* \big/ \{\pm 1\}\right)^3 \cong (\mathbb{R}_{>0})^3 \tag{4.3.8}$$

The moduli space of (isometry classes of) flat metrics is:

$$\mathcal{M}_{\text{flat}}(\mathcal{G}_6) \cong \mathbb{T}(\mathcal{G}_6)/\mathcal{N}_\pi \tag{4.3.9}$$

with $\mathcal{N}_\pi = \mathbb{h}(N(\pi))$, where $N(\pi)$ is the normalizer of π in $\mathrm{Aff}(\mathbb{R}^3)$ and

$$\begin{aligned} \mathbb{h} : \mathrm{Aff}(\mathbb{R}^3) &\to GL(3, \mathbb{R}) \\ (A, v) &\mapsto A \end{aligned} \tag{4.3.10}$$

is the holonomy projection map.

We are interested in computing the moduli space of flat *oriented* metrics, i.e., we do not admit orientation-reversal symmetries. So we need to describe the orientation-preserving piece $N^+(\pi) \leqslant N(\pi)$ and then its image $\mathbb{h}(N^+(\pi))$. The idea is to identify this last group as a subgroup of the *group of affinities* $\mathrm{Aff}(\mathcal{G}_6)$, i.e., the affine automorphisms of $\mathbb{R}^3$ that descend to $\mathcal{G}_6 \cong \mathbb{R}^3/\pi$. The reader who dislikes short exact sequences is encouraged to skip this computation. We start by proving a lemma:

Lemma 4.3.5. *The group of affinities $Aff(\mathcal{G}_6)$ is an extension of $\mathbb{Z}_2$ by $Out(\pi)$.*

Proof. For any flat compact manifold $\mathcal{G}$, the $\mathrm{Diff}(\mathcal{G})$-action on the loop space $\mathcal{L}(\mathcal{G})$ induces, by the Dehn-Nielsen-Baer theorem for flat manifolds, an isomorphism between $Out(\pi_1(\mathcal{G}))$ and the *Mapping Class Group* $\mathbb{M}(\mathcal{G}) := \mathrm{Diff}(\mathcal{G})/\mathrm{Diff}^+(\mathcal{G})$. Moreover, $\mathrm{Aff}(\mathcal{G})/\mathrm{Aff}^+(\mathcal{G}) \cong Out(\pi_1(\mathcal{G}))$ (Theorem 6.1 of [Cha86]).

Now, $\mathrm{Aff}^+(\mathcal{G})$ is isomorphic to $\mathrm{Iso}^+(\mathcal{G})$, the orientation-preserving isometries of the flat metric [KN63], hence it is a torus $(\mathbf{S}^1)^{b_1(\mathcal{G})}$. Since $b_1(\mathcal{G}_6) = 0$, it follows that $\mathrm{Aff}^+(\mathcal{G}_6) \cong \mathbb{Z}_2$. $\square$

The group $Out(\pi)$ fits into an exact sequence:

$$1 \to (\mathbb{Z}_2)^3 \to Out(\pi) \to S_3 \times \mathbb{Z}_2 \to 1 \tag{4.3.11}$$

and the group $N(\pi)$ fits into another exact sequence:

$$1 \to \pi \to N(\pi) \to Out(\pi) \to 1 \tag{4.3.12}$$

The last $\mathbb{Z}_2$ factor in 4.3.11 is an orientation-reversal (it acts as $-\mathbf{1}_{3\times 3}$). So we consider the subgroup $Out^+(\pi) \leqslant Out(\pi)$. This fits into the exact sequence:

$$1 \to (\mathbb{Z}_2)^3 \to Out^+(\pi) \to S_3 \to 1 \tag{4.3.13}$$

and is in fact the *wreath product*, i.e., the semi-direct product $W := (\mathbb{Z}_2)^3 \rtimes S_3$ induced from the permutation action. Thus the desired group is an extension:

$$1 \to \pi \to N^+(\pi) \to W \to 1 \tag{4.3.14}$$

Now, we compute $\mathbb{h}(N^+(\pi))$. This is isomorphic to $N^+(\pi)/Ker(\mathbb{h}|_{N^+(\pi)})$. We look at W and π separately:

- π: recall that this is a crystallographic group and fits into an exact sequence:

$$1 \to \Lambda_\pi \to \pi \to H_\pi \to 1 \tag{4.3.15}$$

  where $\Lambda_\pi \cong \mathbb{Z}^3$ acts as translations and $H_\pi \cong \mathbb{K}$ acts as usual (reflects two of the three coordinates).

- W: the factor $(\mathbb{Z}_2)^3$ acts as translations on $\mathbb{R}^3$, so its action on π is killed by $\mathbb{h}$. The S_3 factor just permutes the coordinates on $\mathbb{R}^3$, and it acts on $\mathbb{K} \cong \pi/\Lambda_\pi$ by permutting the three non-trivial elements.

It follows that:

$$\mathbb{h}(N^+(\pi)) \cong \mathbb{K} \rtimes S_3 \tag{4.3.16}$$

Therefore the moduli space looks like:

$$\mathcal{M}_{\text{flat}}(\mathcal{G}_6) \cong (\mathbb{R}_{>0})^3 \big/ \mathbb{K} \rtimes S_3 \tag{4.3.17}$$

Note that this has a striking similarity to the moduli space of special Lagrangian deformations $\mathcal{M}_{\mathrm{sLag}} \cong \mathbb{R}^3_{\mathbf{Y}} \backslash \mathbf{Y}$ of the smooth fibers of X: in fact, formula 4.3.17 implies that $\mathcal{M}_{\mathrm{flat}}$ is a quotient of $\mathcal{M}_{\mathrm{sLag}}$ by the action of S_3 on $\mathbb{R}^3$ that permuttes the coordinates. We conjecture the reason is the following:

Conjecture 4.3.6. *A deformation of the semi-flat Calabi-Yau metric on $f : X \to \mathbb{R}^3_{\mathbf{Y}}$ preserving the special Lagrangian fibration induces a flat deformation of the central fiber $f^{-1}(0) \cong \mathcal{G}_6$. If a special Lagrangian deformation is induced from a symmetry of the singular set $\mathbf{Y}$, then the flat structure on $\mathcal{G}_6$ is unchanged.*

In the above we only consider deformations that do not lead to Gromov-Hausdorff collapse of the fibers. If this happens, the conjecture should be modified to allow a flat collapse of $\mathcal{G}_6$. It is proved in [BDP17] that the only affine class of flat collapse of $\mathcal{G}_6$ is a two-dimensional flat orbifold known as $\mathbb{RP}(2, 2;)$ - the quotient of a four-punctured $\mathbf{S}^2$ by the antipodal map.

We finish this section by pointing out a very interesting fact: $\mathbb{h}(N_{\mathrm{Aff}(\mathbb{R}^3)}(\pi))$ is the group of symmetries of the *Borromean rings* $\mathcal{R}$ (see Figure 4.4), and $\mathbb{h}(N^+_{\mathrm{Aff}(\mathbb{R}^3)}(\pi))$ the group of symmetries preserving a framing. The connection with our picture is a consequence of the following beautiful result of Zimmerman [Zim90]: $\mathcal{R}$ is the branching locus of a 2-fold covering map $b : \mathcal{G}_6 \to \mathbf{S}^3$ and $\mathbb{h}(N^+_{\mathrm{Aff}(\mathbb{R}^3)}(\pi))$ is given exactly by $\mathrm{Aff}(\mathcal{G}_6)/\mathrm{Gal}(b)$ (where $\mathrm{Gal}(b) \cong S_3$ is generated by the three maps permutting the sheets). Thus, the symmetries of $\mathcal{R}$ must encode the structure of flat deformations of $\mathcal{G}_6$, and assuming Conjecture 4.3.6 is true, also of $\mathcal{M}_{\mathrm{sLag}}(X)$.

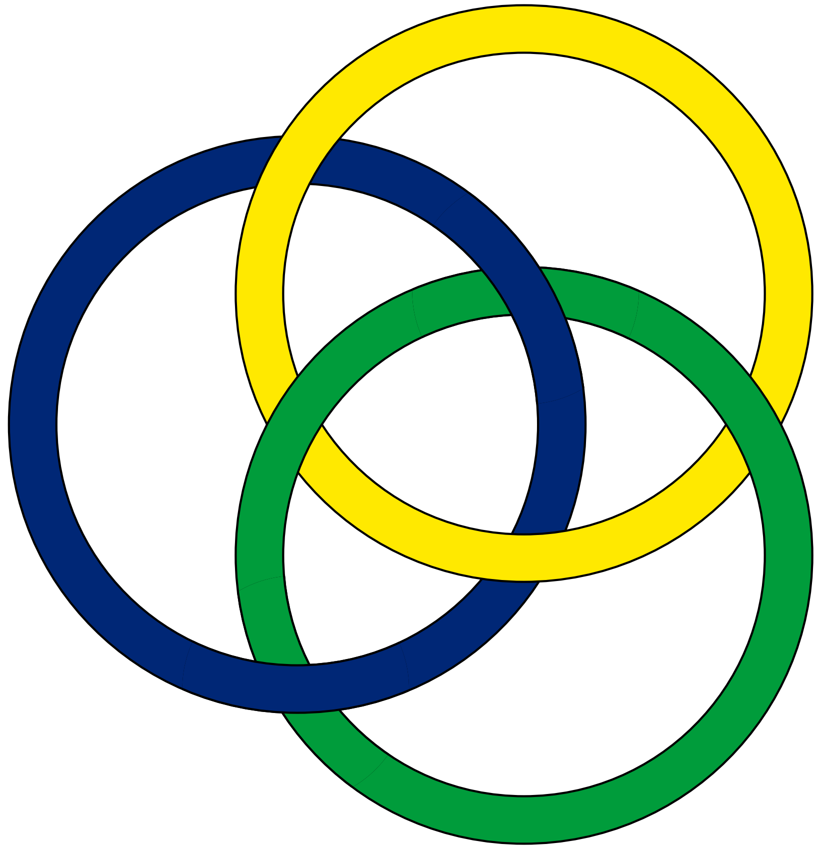

Figure 4.1: Borromean rings

LaTeX code by Dan Drake, available at http://math.kaist.ac.kr/∼drake (colors were changed).

R. H. Fox proved that a branched covering is uniquely determined by its restriction to the unramified locus. Therefore, equivalence classes of n-sheeted branched coverings $c : A \to (B, L)$ are in one-to-one correspondence with equivalence classes of monodromy representations $\rho \in Hom(\pi_1(A \backslash c^{-1}(L)), S_n)$. In our situation, the monodromy of b around each of the three components of $\mathcal{R}$ is given by the permutation exchanging the two sheets. These correspond to the three copies of $\mathbb{Z}_2 \leqslant S_3$. These, of course, generate the whole S_3, which is also the symmetry group of $\mathbb{R}^3_{\mathbf{Y}}$.

4.4 Envisioning a G_2-conifold transition

We now close the circle of ideas from the previous section and give a precise conjecture regarding the relationship between the geometric structures discussed. Let $\mathscr{M}(2,2)$ denote the generic orbit of the Atiyah-Hitchin moduli space of $SU(2)$ monopoles of charge 2. Moreover, in this section we write $H_{\mathcal{G}_6}$ instead of $\mathbb{K}$ to make the connection with $\mathcal{G}_6$ explicit. We compile a few facts:

1. $\mathscr{M}(2,2) \cong SO(3)/H_{\mathcal{G}_6}$. Moreover, $H_{\mathcal{G}_6}$ is the Borel subgroup of $SO(3)$, and the associated complete real flag $\mathcal{F}$ satisfies $\pi_1(\mathcal{F}) = Q_8$, the quaternion group. Since $\mathcal{F}$ is closed and oriented, it is a homology sphere, and is in fact the quotient $\mathbf{S}^3/Q_8$. In particular, it is a spherical space in the sense of Thurston.

2. The base $\mathbb{R}^3_{\mathbf{Y}}$ of the SYZ fibration on $T^*\mathcal{G}_6$ is a decompactification of $SO(3)/H_{\mathcal{G}_6}$.

3. $SO(3)/H_{\mathcal{G}_6}$ is also the orbit space of the $SO(3)$-action by left multiplication on the nearly-Kähler space $SU(3)/U(1)^2$.

4. The metric cone $C(SU(3)/U(1)^2)$ admits a G_2-structure.

5. The Hantzsche-Wendt space $\mathcal{G}_6$ is a double cover of $\mathbf{S}^3$ branched over the Borromean rings $\mathcal{R}$.

6. The complete real flag $\mathcal{F}$ is a double cover of $\mathbf{S}^3$ branched over three fully linked unknots [AW01].

We now imagine that we start with $\mathcal{R}$ on $\mathbf{S}^3$ and take a limit where the three links touch at a single point, which we call a vertex. We then resolve this singularity by deforming the links to be fully linked. At the level of the SYZ fibration, this corresponds to deforming the fiber $\mathcal{G}_6$ creating enhanced monodromy at a point (there is now 8 sheets coming together at the vertex). This monodromy persists in the resolution, which is $\mathcal{F}$. However, we have made a transition from a flat 3-cycle to a non-flat one ($\mathcal{F}$ can not be flat as its fundamental group is finite). At the level of G_2-geometry, we arrive at the following:

Conjecture 4.4.1. *(G_2-conifold transition): There exists a rank 2 complex vector bundle $\mathbf{E}_{\mathcal{F}} \to \mathcal{F}$ such that*

1. *$\mathbf{E}_{\mathcal{F}}$ has a metric with holonomy contained in G_2*
2. *There is a metric degeneration:*

$$\mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6 \rightsquigarrow C(SU(3)/U(1)^2)/\mathbb{Z}_2 \dashleftarrow \mathbf{E}_{\mathcal{F}} \tag{4.4.1}$$

that exchanges the flat 3-cycle $\mathcal{G}_6$ by the spherical 3-cycle $\mathcal{F}$.

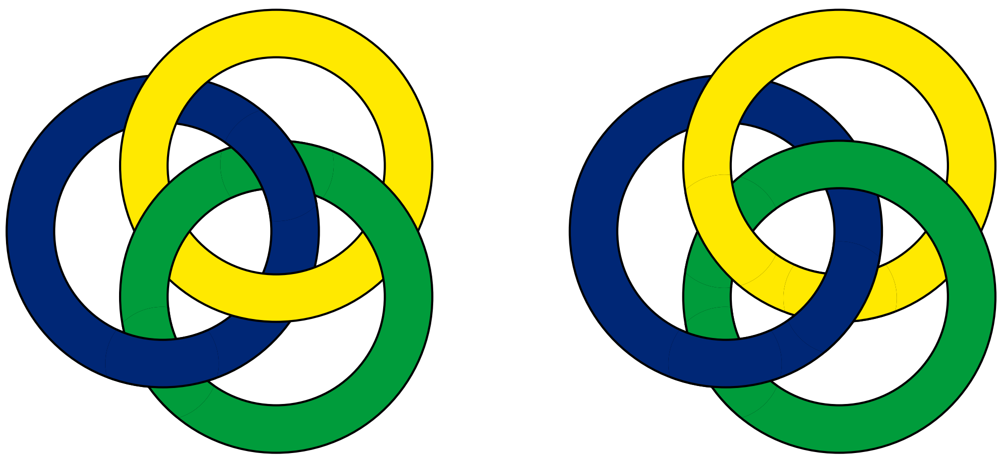

Figure 4.2: Left: Borromean rings. Right: Three fully linked unknots.

Chapter 5

Flat Higgs Bundles and A-branes

5.1 Flat Higgs bundles

We start this chapter by introducing the notion of *flat Higgs bundles*, which are special solutions to the Acharya-Pantev-Wijnholt system 4.2 adapted to platycosms. We will relate it to the character varieties of Bieberbach group of 2.2 using Corlette's theorem.

We are interested in studying Higgs bundles over Bieberbach manifolds. To simplify notation, we will make no distinction between δ and the induced flat connections on other tensor bundles over Q, and will denote all of them by δ. The same will hold for a connection A on a vector bundle over Q and its associated bundles. Moreover, given a flat bundle (E, A), we denote its local system of flat sections by E_A to distinguish it from the *smooth* bundle E. For example, $T_\delta Q$ denotes the sheaves of flat vector fields on Q. We note that even though these sheaves have the same fibers as their smooth counterparts, their stalks contain only flat sections. We will also use the notation $\Omega^1_\delta(Q)$ for the sections of $T^*_\delta Q$.

Remark 5.1.1. All constructions in this chapter are valid in the larger realm of flat *affine* manifolds - i.e., smooth manifolds with a flat connection on the tangent bundle which is not necessarily compatible with a metric. However, we restrict the discussion to Bieberbach manifolds in order to connect directly with the previous chapters.

We recall the following fact: given a smooth manifold N, let $E \to N$ be a complex vector bundle with a hermitian metric h. Then h induces an isomorphism $\overline{E} \cong E^*$. If one is given a hermitian connection A on (E, h), then the covariant derivative $\nabla_A : \Gamma(E) \to \Omega^1(E)$ has an h-adjoint $\nabla_A^* = \overline{\nabla}_A$. This gives another operator $\nabla_A^\dagger : \Gamma(E) \to \Omega^1(E)$ given by $\nabla_A^\dagger := \sigma \circ \overline{\nabla}_A \circ \sigma$, where σ is complex conjugation.

Definition 5.1.2. A *smooth* $GL(r, \mathbb{C})$*-Higgs bundle* on a smooth manifold Q is a tuple (E, h, A, θ) consisting of a complex rank r vector bundle $E \to Q$ with a hermitian metric h, a unitary flat connection $A \in \Omega^1(End(E))$, and a *real* $\mathcal{C}^\infty_Q$-linear bundle map $\theta : \Gamma(E) \to \Gamma(E \otimes T^*Q)$ satisfying the condition:

- $\theta \wedge \theta = 0$

We will say that (E, h, A, θ) is a *flat Higgs bundle* if, furthermore, the following *flatness conditions* are satisfied:

- $\nabla_A \theta = 0$

- $\nabla_A^\dagger \theta = 0$

Flat Higgs bundles are special solutions of the Acharya-Pantev-Wijnholt equations 4.2.7 4.2.8. Those equations should be thought of as analogues of Hitchin's equations in the category of flat smooth vector bundles.

Remark 5.1.3. The reality condition on the Higgs field $\theta \in \Omega^1(End(E))$ means that $\overline{\theta} = \theta$. The dual Higgs field $\overline{\theta}$ can be seen as a map $\overline{\theta} : T(Q) \otimes E \to E$, and therefore it induces an action of the tensor algebra $\mathcal{T}^\bullet(T(Q))$ on E. The condition $\theta \wedge \theta = 0$ means that this action descends to an action of the symmetric algebra $\mathcal{S}^\bullet(T(Q))$ on E.

Notice that a smooth Higgs bundle is flat if and only if the Higgs field θ belongs to the local system $End_A(E) \otimes T^*Q$. If we are given a Riemannian metric δ on Q, there is another local system structure on $End(E) \otimes T^*Q$ given by the connection $A \otimes \delta$. This is a connection that preserves the full metric on the bundle, unlike A. It follows from the product formula for tensor product connections that A-flat Higgs bundles and $A \otimes \delta$-flat Higgs bundles are equivalent if and only if θ takes values in $T^*_\delta Q$, i.e., in flat 1-forms on Q. This is a natural condition when working with flat Riemannian manifolds such as platycosms.

Definition 5.1.4. Let (Q, δ) be a flat Riemannian manifold. A *flat Higgs bundle on* (Q, δ) is a flat Higgs bundle (E, h, A, θ) on Q such that $\theta \in \Omega^1_\delta(End_A(E)) := \Gamma(T^*_\delta Q \otimes End_A(E))$.

Define the *hermitian Hodge-star operator* $\star_h$ on basic elements of $\Omega^1(End(E))$ by $\star_h(A \otimes \alpha) = \overline{A} \otimes \star\alpha$ and extended by linearity.

Lemma 5.1.5. *(E, h, A, θ) is a flat Higgs bundle on (Q, δ) if and only if:*

$$\nabla_{A\otimes\delta}\theta = 0$$

$$\nabla_{A\otimes\delta}(\star_h\theta) = 0 \tag{5.1.1}$$

Proof. Follows from the discussion above and the formula:

$$\nabla^{\dagger}_{A\otimes\delta} = \star_h \circ \nabla_{A\otimes\delta} \circ \star_h \tag{5.1.2}$$

□

The following table compares Higgs bundles in the holomorphic and flat worlds:

Holomorphic	Flat
$(X, \overline{\partial})$ Kähler manifold	(Q, δ) flat Riemannian manifold
$\Omega^{1,0} \hookrightarrow \Omega^1(X)$	$\Omega^1_\delta \hookrightarrow \Omega^1(Q)$
$(E, \overline{\partial}_E : E \to E \otimes \Omega^{0,1})$ holomorphic bundle	$(E, h, \nabla_A : E \to E \otimes \Omega^1_\delta)$ hermitian flat bundle
$\phi : E \to E \otimes \Omega^{1,0}$	$\theta : E \to E \otimes \Omega^1_\delta$
$\phi \wedge \phi = 0$, $\overline{\partial}^2_E = 0$, $\overline{\partial}_E\phi = 0$ (F-terms)	$\theta \wedge \theta = 0$, $\nabla^2_A = 0$, $\nabla_A\theta = 0$ (F-terms)
$\overline{\partial}_E\phi^\dagger = 0$ (D-term)	$\nabla^\dagger_A\theta = 0$ (D-term)

Remark 5.1.6. When solving Hitchin's equations for (holomorphic) Higgs bundles, there are two different perspectives: in the *holomorphic perspective*, one fixes a hermitian metric and solves for a connection and a Higgs field, with the holomorphic structure coming from the $(0,1)$ part of the connection; in the *hermitian perspective*, we fix a holomorphic structure on the bundle and solve Hitchin's equations for a hermitian metric and a Higgs field, with respect to the unitary Chern connection. As we are fixing the flat structure on Q, our current formulation of flat Higgs bundles is closer to the latter. However, one notices that this duality also works in our case; this is clear from the Acharya-Pantev-Wijnholt equations, and is reminiscent from our discussion on ADE G_2-platyfolds in the first chapter, when there was a similar duality between flat structures on the base and on the fibers.

To explain the meaning of the D-term, recall the following important result [Don87], [Cor88]:

Theorem 5.1.7. (Corlette, Donaldson): *Let G_c be a semisimple algebraic group and K a maximal compact subgroup. Let (Q,g) be a compact Riemannian manifold with fundamental group π, and let $(\widetilde{Q},\widetilde{g})$ be its universal cover. Fix a homomorphism $\rho:\pi\to G_c$ and let $h:\widetilde{Q}\to G_c/K$ be a ρ-equivariant map. Then the following are equivalent:*

1. *$h:\widetilde{Q}\to {}^{G_c}/_{K}$ is a harmonic map of Riemannian manifolds*
2. *The Zariski closure of $\rho(\pi)$ is a reductive subgroup of G_c (i.e., ρ is semisimple)*

Moreover, if ρ is irreducible, the harmonic map is unique.

In the language of flat bundles, the theorem says that the gauge orbit of a flat, stable G_c-connection has a unique harmonic metric. We will now explain this more carefully, as this is one of the crucial points of the theory. The reader familiar with harmonic metrics is encouraged to skip the next paragraphs.

This theorem is a manifestation of a deep relationship between algebraic and symplectic geometry, relating stability of orbits of G_c-actions to zeros of moment maps. In its simplest form, this is expressed by the Kempf-Ness theorem [KN79]: let V be a complex vector space and $\rho : G_c \to GL(V, \mathbb{C})$ a representation. Then a G_c-orbit $G_c v$ is stable $\Leftrightarrow$ $G_c v$ has maximal dimension and contains a shortest vector v_0. On the symplectic side, one picks a hermitian form h on V endows V with a symplectic structure ω, and there is an induced action of the compact subgroup $K \subset G_c$ preserving this form. The data (h, ω, K) gives a moment map μ, and $\mu(v_0) = 0 \Leftrightarrow v_0$ is the shortest vector in $G_c v$.

The Corlette-Donaldson theorem is an infinite-dimensional analogue of this result. The dictionary goes as follows: V is replaced by a flat G_c-bundle $E \to (Q, g)$, ρ is now the monodromy representation of a flat connection D, and G_c is replaced by the group of G_c-gauge transformations $\mathscr{G}_c$. The algebraic point of view now says that D is irreducible $\Leftrightarrow$ one can find in its orbit $\mathscr{G}_c(D)$ a "shortest" metric - i.e., harmonic. The symplectic point of view sheds more light in the harmonicity condition: once one picks *any* hermitian metric h on the bundle, h and g together induce a Kähler metric ω on the space of connections $\Omega^1(Q, Ad(E))$. The group of unitary gauge transformations $\mathscr{G}_h$ preserves ω. Write $D = A + \theta$, where A preserves ω and θ is hermitian. The associated moment map is $\mu = \nabla_A^\dagger \theta$, and its vanishing is equivalent to a harmonic metric. Thus, a solution to the D-term is a harmonic metric.

The reason for the name "harmonic" is the following: a metric on $(E, D) \to Q$ is a choice of inner product at every fiber, compatible with the flat structure D. The pullback of (E, D) to $\widetilde{Q}$ is a trivial flat bundle, and the same is true for the bundle of metrics, which is $\widetilde{Q} \times SL(n, \mathbb{C})/SU(n)$. A metric on $(E, D) \to Q$ is then just a π-equivariant section s of this bundle, and we say the metric is harmonic if s is a harmonic map, where the symmetric space $\mathscr{S} := SL(n, \mathbb{C})/SU(n)$ is given its canonical Riemannian structure. Now, Corlette's proof of theorem 5.1.7 [Cor88] shows that, in terms of the section s, one can write $\theta = ds$ and $\nabla_A = s^*\nabla_{\mathscr{S}}$, where $\nabla_{\mathscr{S}}$ is the Levi-Civita connection of $\mathscr{S}$. Thus the harmonicity condition is exactly what one expects:

$$\nabla_A^\dagger \theta = \star s^* \nabla_{\mathscr{S}} \star d(s) = 0 \tag{5.1.3}$$

where the trivial flat connection d can be thought as the Levi-Civita connection for the flat metric on $\widetilde{Q}$.

We note also that Proposition 2.2 of [Cor88] shows that the vanishing of the D-term is equivalent to minimizing $||\theta||^2_{L^2}$. Thus, harmonic metrics are those such that the decomposition $D = A + \theta$ gives the shortest Higgs field. This completes the analogy with the Kempf-Ness theorem.

In the holomorphic category, Simpson [Sim92] used theorem 5.1.7 to show that a harmonic metric on a holomorphic bundle over a Kähler manifold gives a holomorphic Higgs bundle. We will show an analogous result in the flat world: a harmonic metric on a flat bundle over a compact Riemannian manifold Q is given by a solution to the APW system. One would expect, moreover, that when Q is a Bieberbach manifold, such solutions are given by flat Higgs bundles (however, see the discussion below).

We say (E, D) is a *flat reductive bundle* if D has reductive monodromy.[1] The following is the analogue of Simpson's theorem on the flat category:

Theorem 5.1.8. *For a compact Riemannian manifold Q, there is a bijection between the following data:*

$$\textit{Flat reductive bundles } (E, D) \longleftrightarrow \textit{Solutions } (E, h, A, \theta) \textit{ of the APW system}$$

Proof. Suppose first we are given a flat reductive bundle (E, D) on Q. Pick a hermitian metric g on E. Then there is a unique C^∞-linear $\theta \in \Omega^1_\delta(End(E))$ satisfying $\overline{\theta} = \theta$ (i.e., θ is real with respect to g) and such that $\nabla_A = \nabla_D - \theta$ satisfies:

$$\nabla_A g = 0 \tag{5.1.4}$$

The Corlette-Donaldson theorem says there is a π-equivariant harmonic map $h : \widetilde{Q} \to SL(n, \mathbb{C})/SU(n)$ associated to (∇_A, θ). The harmonicity condition is

[1]More precisely, if the Zariski closure of the image of the monodromy representation is a reductive subgroup of G_c.

$$\nabla_A^\dagger \theta = 0 \tag{5.1.5}$$

where conjugation is taken with respect to h. Together, equations 5.1.4 and 5.1.5 imply that $\nabla_A \theta = 0$. Therefore, $\nabla_A^2 - \theta \wedge \theta = \nabla_D^2 = 0$, and (E, h, A, θ) a solution of the Acharya-Pantev-Wijnholt system.

For the converse, start with a solution of APW (E, h, A, θ) and let $\nabla_D = \nabla_A + \theta$. Then:

$$\begin{aligned}
\nabla_D^2 &= (\nabla_A + \theta) \circ (\nabla_A + \theta) \\
&= \nabla_A^2 + \nabla_A \circ \theta + \theta \circ \nabla_A + \theta \wedge \theta \\
&= \nabla_A^2 + \nabla_A \theta + \theta \wedge \theta \\
&= 0
\end{aligned}$$

It remains to show that D has reductive monodromy. Let ρ_D and ρ_A be the monodromy representations of D and A, respectively. Then $\rho_D(\pi) \subseteq \rho_A^{\mathbb{C}}(\pi)$. But A is unitary, hence $\rho_A(\pi) \subseteq U(n)$ is compact. Thus $\rho_A^{\mathbb{C}}(\pi)$ is reductive and so is $\overline{\rho_D(\pi)}$. $\square$

Remark: One can define an appropriate notion of morphism for flat Higgs bundles. Then the bijection extendeds to an equivalence between the subcategory of local systems with reductive monodromy and the category of flat Higgs bundles. We omit the proof since we only care about moduli spaces of objects here.

One could ask what happens to Theorem 5.1.8 when (Q, δ) is a Bieberbach manifold. In view of the results of Chapter 3, it seems reasonable to expect that all solutions of the Acharya-Pantev-Wijnholt system on (Q, δ) are given by flat Higgs bundles (i.e., $F_A = \theta \wedge \theta = 0$). For the $SL(2, \mathbb{C})$ case, this conjecture holds if and only if the following is true: given a harmonic map $f : (\mathbb{R}^3, d_{\mathbb{R}^3}) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$, where $d_{\mathbb{R}^3}$ is the flat metric and $d_{\mathbb{H}^3}$ is the constant curvature -1 metric, then the pullback $f^*\nabla_{d_{\mathbb{H}^3}}$ of the Levi-Civita connection of $d_{\mathbb{H}^3}$ is a flat connection. The author sees no reason why this result should be true. This is related to the question of whether there exist "exotic" G_2-deformations of ADE G_2-platyfolds - i.e., deformations that do not preserve the coassociative fibration structure, and hence not obtainable through Donaldson data and unfolding of ADE singularities.

5.2 The moduli space of A-branes on $\mathcal{G}_6$

We now return to our main example, the Hantzsche-Wendt Calabi-Yau $X = T^*\mathcal{G}_6$. Proposition 5.1.8 shows that a configuration of n A-branes wrapping $\mathcal{G}_6$ is given by specifying a reductive flat $SL(n, \mathbb{C})$-connection on a vector bundle over $\mathcal{G}_6$. Hence, $\mathcal{M}_{\mathrm{IIA}}(\mathcal{G}_6)$ is the character variety $\mathrm{Char}(\mathcal{G}_6, SL(n, \mathbb{C}))$. In what follows, X_n denotes the data of the Calabi-Yau space X together with n A-branes wrapping $\mathcal{G}_6$. That is, X_n is the orbifold $T^*\mathcal{G}_6/\mathbb{Z}_n$ with a $\mathbb{Z}_n$-singularity at $\mathcal{G}_6$. Here, $\mathbb{Z}_n \subset SO(3)$ acts on each fiber via the fundamental representation.

We let π be again the Hantzsche-Wendt group, with $1 \to \mathbb{Z}^3 \to \pi \to \mathbb{K} \to 1$, where $\mathbb{K} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. From section 2.2.1, we know that the character variety of the three-torus $\mathbf{T}$ is given by:

$$\mathrm{Char}(\mathbb{Z}^3, SL(n,\mathbb{C})) \cong (\mathbb{C}^*)^{3n-3}/\Sigma_n = \prod_{i=1}^{3} \left((\mathbb{C}^*)^{n-1}/\Sigma_n\right) \tag{5.2.1}$$

where Σ_n acts by permutations on $(\mathbb{C}^*)^{n-1} \cong \{z_1 z_2 \dots z_n = 1\} \subset (\mathbb{C}^*)^n$.

In section 2.2, we determined that there is a map $r : \mathrm{Char}(\pi, SL(n,\mathbb{C})) \to \mathrm{Char}(\mathbb{Z}^3, SL(n,\mathbb{C}))$. Moreover, there is a $\mathbb{K}$-action on this last space, given in terms of the presentation 5.2.1 by

$$(i,j)[(z_1, z_2, z_3)] = [(z_1^i, z_2^j, z_3^{ij})] \tag{5.2.2}$$

where $i, j \in \{\pm 1\}$.

The main result in section 2.2 was that $Im(r)$ is contained in $\mathrm{Fix}(\mathbb{K})$. The image determines $\mathrm{Char}(\pi, SL(n,\mathbb{C}))$ possibly up to a finite cover given by non-trivial representations of $\mathbb{K}$ mapping to the same element of $\mathrm{Hom}(\pi, SL(n,\mathbb{C}))$.

When $n = 2$ it is easy to describe $\mathrm{Fix}(\mathbb{K})$. Let $\alpha = (1, -1)$. Then:

$$\mathit{Fix}(\alpha) = [(\pm 1, \mathbb{C}^*, \mathbb{C}^*)] \cup [(\mathbb{C}^*, \pm 1, \pm 1)] = (\mathbb{C}^2)_{z_2, z_3} \cup \mathbb{C}_{z_1} \tag{5.2.3}$$

Notice that the first factor is contributed by $-1 \in \mathbb{Z}_2$ and the second by $1 \in \mathbb{Z}_2$.

We can play a similar game for the other two non-trivial elements of $\mathbb{K}$. Hence:

$$\mathrm{Fix}(\mathbb{K}) = \bigcap_{(i,j,k) \in \langle (1,2,3) \rangle} \left((\mathbb{C}^2)_{z_i, z_j} \cup \mathbb{C}_{z_k}\right) = \mathbb{C}_{z_1} \cup \mathbb{C}_{z_2} \cup \mathbb{C}_{z_3} \tag{5.2.4}$$

Thus Fix($\mathbb{K}$) is a bouquet of three complex lines touching at a point. The image $Im(r)$ can be computed directly from a presentation of π to show that r is in fact surjective; essentially, this is because an element in the bouquet, say $(a, 0, 0)$, is a representation which is non-trivial only at a single generator, so will be automatically a representation of π. Thus:

$$\mathrm{Char}^0(\pi, SL(2, \mathbb{C})) \cong \mathbf{Y}_{\mathbb{C}} := \mathbb{C}_{z_1} \cup \mathbb{C}_{z_2} \cup \mathbb{C}_{z_3} \tag{5.2.5}$$

We recall that the notation $\mathbf{Y}_{\mathbb{C}}$ means that the space looks like a trident of $\mathbb{C}$'s touching at the origin. The "real version" $\mathbf{Y}$ of $\mathbf{Y}_{\mathbb{C}}$ has appeared before as the discriminant locus of the SYZ fibration $f : X \to \mathbb{R}^3_{\mathbf{Y}}$. The space $\mathbf{Y}_{\mathbb{C}}$ will also appear again in section 6.3 when we discuss the moduli space of B-branes in the mirror $\widehat{X}$.

Remark 5.2.1. Formula 5.2.5 only computes the connected component of the trivial representation. Up to conjugation, there are also 3 other non-trivial representations of $\mathbb{K}$ that map to the trivial representation of π.

Now here is the important point: we can actually use this to verify M-theory/IIA duality - i.e., the equivalence between the moduli space of vacua in the two theories - in this case. Recall that the M-theory dual of X_2 is the Hantzsche-Wendt G_2-platyfold $M = \mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6$. From the discussion in Chapter 3, we have that:

$$\mathcal{M}_{G_2}(M) = H^0_{\mathrm{flat}}(\mathcal{G}_6, X_2 \otimes \mathfrak{u}(1)) \tag{5.2.6}$$

So we need to find the flat sections of the orbifold X_2. These are given by elements in $\Omega^1_\delta(\mathcal{G}_6)$ fixed by the monodromy $H_{\mathcal{G}_6} \cong \mathbb{K}$. Recall that the fixed set of the monodromy action of $\mathbb{K}$ on $T^*_q\mathcal{G}_6 \cong \mathbb{R}^3$ is the trivalent vertex $\mathbf{Y} \subset \mathbb{R}^3$. If we now complexify this moduli space by adding C-fields (parametrized by their holonomies, which takes values in $U(1)$), we also obtain a trident of $\mathbb{C}$'s touching at a point. We have established:

Theorem 5.2.2. *(Physical version): The moduli space of M-theory vacua on M is homeomorphic to the moduli space of two A-branes wrapping the zero-section of X.*

(Mathematical version): The moduli space of complexified G_2-structures on the Hantzsche-Wendt G_2-platyfold $\mathbb{C}^2/\mathbb{Z}_2 \times_{\mathbb{K}} \mathcal{G}_6$ is homeomorphic to the $SL(2,\mathbb{C})$-character variety of the Hantzsche-Wendt Calabi-Yau $T^*\mathcal{G}_6$:

$$\mathcal{M}^{\mathbb{C}}_{G_2}(M) \cong \mathcal{M}_{IIA}(X) \tag{5.2.7}$$

Remark 5.2.3. We mentioned before that there are three other connected components of $\mathrm{Char}(\pi, SL(2,\mathbb{C}))$ given by three isolated points. They correspond up to conjugation to the three representations of $\mathbb{K}$ that map to $0 \in Hom(\pi, SL(2,\mathbb{C}))$. Under duality, supposedly these three points correspond to rigid G_2-structures on M, or at least G_2-structures that do not admit deformations preserving the structure of a coassociative fibration.

The M-theory moduli space for this particular example could have been computed without referring to the results of chapter 3, as we now explain. This computation first appeared on Joyce's book [Joy00], pages 317-319. The possible ways to smooth the singularity $\mathbb{C}^2/\mathbb{Z}_2$ is to either resolve it with a blow-up or deform it. There are two families of deformations, given by

$$\begin{gathered}\left\{(z_1, z_2, z_3, \epsilon); z_1^2 + z_2^2 + z_3^2 = \epsilon\right\} \subset \mathbb{C}^3 \times \mathbb{R}_{>0} \\ \left\{(z_1, z_2, z_3, \epsilon); z_1^2 + z_2^2 + z_3^2 = \epsilon\right\} \subset \mathbb{C}^3 \times \mathbb{R}_{<0}\end{gathered} \tag{5.2.8}$$

Moreover, the resolved family is parametrized by the volume of the blown-up $\mathbb{P}^1$. So we once again have a trivalent vertex $\mathbf{Y}$, which we again complexify. The result follows now because once we have resolved one fiber $\mathbb{C}^2/\mathbb{Z}_2$, the hyperkähler structure in all other fibers are fixed up to volume, since each Kähler class lies in $\mathfrak{u}(1)$, and flatness of the vertical section $\underline{\omega}$ fixes the volumes once and for all.

The description of the moduli space as a character variety allows us to go beyond and generalize the computation to include more branes. For higher n, we would need a generalization of formula 5.2.4 characterizing $Im(r)$.

Remark 5.2.4. We will address this last point, however we must warn the reader that the content of this remark is not rigorously developed, and is included just in order to suggest how the computation should go in the general case. We get back to the main track on Proposition 5.2.5.

We conjecture the following formula for $\mathrm{Fix}(\mathbb{K})$ (which should allow us to characterize $Im(r)$ in general):

$$\mathrm{Fix}(\mathbb{K}) \cong \bigcup_{K, K' \leqslant \mathbb{K}} \left(\left(T^3 \big/ K\right)^{\Sigma_n} \cap \left(T^3 \big/ K'\right)^{\Sigma_n} \right) \tag{5.2.9}$$

where the union is over distinct proper subgroups of $\mathbb{K}$.

For $n = 2$ a simple calculation shows that the intersecting factors are perpendicular $\mathbb{C}^2$'s, resulting in an union of three $\mathbb{C}$'s as in formula 5.2.4. Hence, formula 5.2.9 is correct for $n = 2$.

We will illustrate the computation of $\mathrm{Fix}(\mathbb{K})$ for $n = 3$ using formula 5.2.9. Already in this case, the computation becomes considerably more intricate. For each $\alpha \in \mathbb{K}$ let $\overline{\mathrm{Fix}}(\alpha)$ denote the fixed set of α acting on $(\mathbb{C}^*)^{3n-3}$ (i.e., *before* taking the quotient by Σ_n). The formula for the fixed set modulo Σ_n, $\mathrm{Fix}(\mathbb{K})$, is then:

$$\mathrm{Fix}(\mathbb{K}) = \bigcap_{\alpha \in \mathbb{K}} \bigcup_{\sigma \in \Sigma_n} \sigma(\overline{\mathrm{Fix}}(\alpha)) \tag{5.2.10}$$

Fortunately, the calculation is slightly simpler than it looks like, as elements in the same conjugacy class behave very similarly. For a fixed element of $\mathbb{K}$, say $\alpha = (1, -1)$, we get the following contributions:

- **Type I:** these come from the trivial permutation:

$$[\pm 1, (\mathbb{C}^*)^2_{y_1,y_2}, \pm 1] = \mathrm{Sym}^3((\mathbb{C}^*)^2_{y_3=1/y_1y_2})$$

- **Type II:** these come from 2-cycles, as follows: let ϵ be a generator of μ_3, the group of 3rd roots of unity, and $\mathbf{E} := \{(\epsilon^i, \epsilon^i, \epsilon^i); i = 0, 1, 2\}$. Then:

  – $[\mathbb{C}^*_{x_1=x_2^{-1}}, \mathbb{C}^*_{y_1=y_2}, \mathbb{C}^*_{z_1=z_2^{-1}}] = \mathrm{Sym}^3(\mathbf{X}_{x_1,x_2}) \times \mathbf{E} \times \mathrm{Sym}^3(\mathbf{X}_{z_1,z_2})$

– $[\mathbb{C}^*_{x_1=x_3^{-1}}, \mathbb{C}^*_{y_1=y_3}, \mathbb{C}^*_{z_1=z_3^{-1}}] = \mathrm{Sym}^3(\mathbf{X}_{x_1,x_3}) \times \mathbf{E} \times \mathrm{Sym}^3(\mathbf{X}_{z_1,z_3})$

– $[\mathbb{C}^*_{x_2=x_3^{-1}}, \mathbb{C}^*_{y_2=y_3}, \mathbb{C}^*_{z_2=z_3^{-1}}] = \mathrm{Sym}^3(\mathbf{X}_{x_2,x_3}) \times \mathbf{E} \times \mathrm{Sym}^3(\mathbf{X}_{z_2,z_3})$

where we are using the notation:

$$\mathrm{Sym}^3(\mathbf{X}_{a_i,a_j}) := \mathrm{Spec}\big(\mathbb{C}[a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}]^{\Sigma_3}/(a_i a_j, a_k^{-1} a_i a_j)\big)$$

is the Σ_3-quotient of the space $\mathbf{X} = \mathbb{C}^*_{a_i} \cup \mathbb{C}^*_{a_j}$ consisting of two copies of $\mathbb{C}^*$ touching at a point.

- **Type III:** these come from 3-cycles. They contribute with points given by $[\pm 1, e^{k\pi i/3}, \pm 1]$, all of which are contained in the sets from the 2-cycles.

The final answer is then given by intersecting types I and II contributions from different elements of $\mathbb{K}$. We get a union of all $\mathrm{Sym}^3(\mathbf{X}_{a_i,a_j})$. To simplify matters, denote $\mathcal{S}_x = \bigcup \mathrm{Sym}^3(\mathbf{X}_{x_i,x_j})$ and similarly for the other variables. Then

$$Im(r) = \mathcal{S}_x \cup \mathcal{S}_y \cup \mathcal{S}_z \tag{5.2.11}$$

Notice that the analogue of $\mathrm{Sym}^3(\mathbf{X}_{a_i,a_j})$ in the $n = 2$ case would be $\mathrm{Sym}^2(\mathbb{C}^*) \cong \mathbb{C}$, and there would be just one of it. Hence when taking the union over the 3 variables we recover the previous result.

It is natural to conjecture that for general n, $Fix(\mathbb{K})$ looks like a $\mathbf{Y}$-shaped join of three copies of $\bigcup \mathrm{Sym}^n(\mathcal{X})$ for some complicated scheme $\mathcal{X}$.

However, it is clear that the combinatorial complexity of the problem increases rapidly. Basically, for each element of $\mathbb{K}$, we need to calculate the contributions of each cycle type and weigh it with the cardinality of the conjugacy class. Then we need to calculate the intersection of the contributions of the three nontrivial elements of $\mathbb{K}$.

The complications associated with the computations in the previous remark led us to develop a different approach towards computing the character variety. This approach will turn out to have a remarkable connection to the mirror symmetry story to be developed in the next chapter. The key point is the following result:

Proposition 5.2.5. *Let T be a maximal torus for $SL(n,\mathbb{C})$, and $W := N(T)/T \cong \Sigma_n$ acting on $N(T)$ by conjugation. Let $\mathcal{C} = \{z_1 z_2 \dots z_n = 1\} \subseteq \big((\mathbb{C}^*)^3\big)^n$. Let $\mathbb{K}$ act on $\mathcal{C}$ by restriction from $(\mathbb{C}^*)^{3n}$, and Σ_n act on $\mathcal{C}/\mathbb{K}$ by restriction from the permutation action on $\big((\mathbb{C}^*)^3/\mathbb{K}\big)^n$. Then the following holds:*

$$\left(T/\Sigma_n\right)^3 \Big/ \mathbb{K} \cong \left(\mathcal{C}/\mathbb{K}\right) \Big/ \Sigma_n$$

Proof. Let $\mathbf{t} = (t_1, t_2, t_3) \in T^3$. Let $\sigma \in W$ and $\alpha = (-1, 1, -1) \in \mathbb{K}$. Then:

$$\begin{aligned}\alpha\circ\sigma(\mathbf{t}) &= \alpha\left(\sigma t_1\sigma^{-1}, \sigma t_2\sigma^{-1}, \sigma t_3\sigma^{-1}\right)\\ &= \left((\sigma t_1\sigma^{-1})^{-1}, \sigma t_2\sigma^{-1}, (\sigma t_3\sigma^{-1})^{-1}\right)\\ &= \left(\sigma t_1^{-1}\sigma^{-1}, \sigma t_2\sigma^{-1}, \sigma t_3^{-1}\sigma^{-1}\right)\\ &= \sigma\left(t_1^{-1}, t_2, t_3^{-1}\right)\\ &= \sigma\circ\alpha(\mathbf{t})\end{aligned}$$

This shows that $(T/W)^3/\mathbb{K} \cong (T/\mathbb{K})^3/W$. To finish, we show that $(T/\mathbb{K})^3 \cong \mathcal{C}/\mathbb{K}$. Write $t_i = (t_i^1, \ldots, t_i^n)$ with $t_i^n = -(t_i^1 + \ldots + t_i^{n-1})$. The key idea is that $\bar{\mathbf{t}}$ can be seen both as an element of $(T/\mathbb{K})^3$ via $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \bar{t}_3)$ and of $\mathcal{C}/\mathbb{K}$ via $\bar{\mathbf{t}} = \left(\bar{t}^1, \ldots, \bar{t}^n\right)$. Then:

$$\begin{aligned}\alpha(t_1, t_2, t_3) &= \left(t_1^{-1}, t_2, t_3^{-1}\right)\\ &= \left((-t_1^1, \ldots, -t_1^n), (t_2^1, \ldots, t_2^n), (-t_3^1, \ldots, -t_3^n)\right)\\ &\cong \left((-t_1^1, t_2^1, -t_3^1), \ldots, (-t_1^n, t_2^n, -t_3^n)\right)\\ &= \alpha\left((t_1^1, t_2^1, t_3^1), \ldots, (t_1^n, t_2^n, t_3^n)\right)\end{aligned}$$

This means that the actions coincide, hence the orbit spaces are the same. □

Notice that[2] $\mathcal{C}$ can be identified with $(\mathbb{C}^*)^{3n-3}$, so we can write more schematically:

[2] This is only true if we ignore the degree grading on the function field, i.e., the fact that we are quotienting by a degree $n-1$ variable.

$$\left((\mathbb{C}^*)^{n-1}/\Sigma_n\right)^3 \Big/ \mathbb{K} \cong \left((\mathbb{C}^*)^3/\mathbb{K}\right)^{n-1} \Big/ \Sigma_n \tag{5.2.12}$$

Also note that:

$$(\mathcal{C}/\mathbb{K}) \Big/ \Sigma_n \hookrightarrow \mathrm{Sym}^n\big((\mathbb{C}^*)^3/\mathbb{K}\big)$$

For a quasiprojective variety X and G a finite subgroup of automorphisms, the quotient X/G is also a quasiprojective variety. In particular, $\mathrm{Sym}^n(X)$ is quasiprojective. Recall there is a *Hilbert-Chow morphism*

$$\mathrm{Hilb}^n(X) \to \mathrm{Sym}^n(X) \tag{5.2.13}$$

where the *Hilbert scheme of n points* $\mathrm{Hilb}^n(X)$ is a reduced scheme parametrizing length n subschemes of X. The morphism sends a length n subscheme to its support cycle in X.

Now, $Im(r)$ consists of those points fixed by the $\mathbb{K}$-action. Thus, we are not interested in all orbits of $\mathbb{K}$, but only those consisting of single points. Under the above isomorphism, these correspond to $\mathbb{K}$-orbits fixed by the Σ_n-action. That is, we are interested in length n subschemes supported at a point of $[(\mathbb{C}^*)^3/\mathbb{K}]$, i.e., the *punctual Hilbert scheme* $\mathrm{PHilb}^n([(\mathbb{C}^*)^3/\mathbb{K}])$. So we expect 5.2.12 to restrict to an isomorphism:

$$\left(((\mathbb{C}^*)^{n-1}/\Sigma_n)^3\right)^{\mathbb{K}} \leftrightarrow \mathrm{PHilb}^n([(\mathbb{C}^*)^3/\mathbb{K}]) \tag{5.2.14}$$

The space on the LHS is just $\mathrm{Char}^0(\mathcal{G}_6, SL(n,\mathbb{C}))$. In the next section, we will show that is not exactly an isomorphism, but that indeed the two spaces are very similar. For example, we will show that for $n = 2$ the RHS is a certain "projectivization" of the LHS[3]. Moreover, in the next chapter we argue in favor of relation 5.2 by constructing the Calabi-Yau mirror $X^\vee$ of X and showing that its *moduli space of mirror B-branes* is:

$$\mathcal{M}_{\mathrm{IIB}} \cong \mathrm{PHilb}^n([(\mathbb{C}^*)^3/\mathbb{K}]) \tag{5.2.15}$$

Thus, formula 5.2 is merely expressing the expected mirror relation:

$$\mathcal{M}_{\mathrm{IIA}}(X) \cong \mathcal{M}_{\mathrm{IIB}}(\widehat{X}) \tag{5.2.16}$$

See Chapter 6 for more on this.

Remark 5.2.6. This result can be generalized to other complex simple Lie groups by using the explicit description of the moduli space of commuting triples by Borel, Friedman and Morgan [BFM02]. In general, this moduli space has multiple components and in each one of them, one needs to work with the maximal tori and Weyl group adapted to the centralizer of the commuting triple.

[3] Intuitively, it seems that the Hilbert scheme "does not see" the rigidity of the three isolated representations, and instead they appear as points at infinity that compactify the three legs of $\mathbf{Y}_\mathbb{C}$ forming a trident of $\mathbb{P}^1$'s.

5.3 Flat Spectral Correspondence

In this section we prove Theorem 5.3.7, a spectral correspondence for flat Higgs bundles. This is a bijection between flat Higgs bundles (E, h, A, θ) on Q and objects we call "flat spectral data". The reason this reformulation is useful is the following: in chapter 4, we proved that M-theory/IIA duality establishes an equivalence between the moduli space $\mathcal{M}^{\mathbb{C}}_{G_2}$ of complexified G_2-structures $\varphi_{\mathbb{C}}$ on an ALE G_2-orbifold $M \to Q$ and the moduli of solutions to the Acharya-Pantev-Wijnholt system. The hermitian metric solving this system is necessarily harmonic, so the Corlette-Donaldson theorem and Theorem 5.1.8 imply that solutions are parametrized by the moduli of flat Higgs bundles (E, h, A, θ) on Q. However, the dependence of flat Higgs data on $\varphi_{\mathbb{C}} = \varphi + iC$ is not compatible with the real structure $\mathcal{M}_{G_2} \subset \mathcal{M}^{\mathbb{C}}_{G_2}$. The spectral data fixes this situation: the flat spectral cover $\widetilde{Q} \to Q$ depends solely on φ, i.e. deformations of φ are completely recovered from flat deformations of $\widetilde{Q} \to Q$. This is in agreement with the main result of chapter 3. The advantage of this new point of view is that now the spectral covers have a clear geometric interpretation in terms of *eigenvalues* of the Higgs field, allowing us to produce G_2-deformations concretely.

We start with a basic example. Let V be a complex vector space of dimension n and $\phi \in End(V)$. If ϕ is diagonalizable, it can be reconstructed by giving its eigenvalues $\{\lambda_1, \ldots \lambda_n\}$, the decomposition of V into ϕ-eigenlines $V = L_1 \oplus \ldots \oplus L_n$ and a matching map $m : L_i \mapsto \lambda_i$. We refer to $(\underline{\lambda}, \underline{L})$ as the *spectral data* associated to (V, ϕ).

Let $p_i(\phi)$ be the coefficient of λ^{n-i} in the expansion of $\det(\lambda \mathbf{1} - \phi) \in \mathbb{C}[t]$. Consider the map:

$$f : End(V) \longrightarrow \mathbb{C}^n \tag{5.3.1}$$

$$\phi \longmapsto (p_1(\phi), \ldots, p_n(\phi)) \tag{5.3.2}$$

Then it is clear that the eigenvalues $\{\lambda_1, \ldots \lambda_n\}$ of ϕ depends only on $h(\phi)$. The map f is a prototype of the Hitchin map 5.3.8 defined below.

Now, let Q be a manifold, and $E \to Q$ a rank n complex vector bundle. Suppose $\phi \in \Gamma(Q, End(E))$. Then to each $\phi_q : E_q \to E_q$ we can associate its spectral data $(\underline{\lambda}_q, \underline{L}_q)$. We think of ϕ as a "twisted family" of endomorphisms parametrized by Q.

As $(\underline{\lambda}_q, \underline{L}_q)$ varies over Q, it defines:

- a subvariety of $Q \times \mathbb{C}$:

  $$\begin{aligned} \widetilde{Q} &= \{(q, \lambda); \lambda \text{ is an eigenvalue of } \phi_q\} \\ &= \left\{(q, \lambda); \det(\lambda \mathbf{1}_{E_q} - \phi_q) = 0\right\} \end{aligned}$$

  called the *spectral cover* of Q associated to ϕ. It comes equipped with a generically $n : 1$ covering map $\pi : \widetilde{Q} \to Q$.

– If ϕ is *generic* - i.e., diagonalizable with distinct eigenvalues at every point - then π is unramified. The same is true if ϕ is *regular*, i.e., has one eigenvalue per Jordan block at every point.

– If ϕ admits more than one eigenline per eigenvalue, then its *ramification locus* is given by:

$$\Delta_\pi = \{q \in Q | \phi_q \text{ has a multiple eigenvalue}\}$$

- A *spectral line bundle*:

$$m : \mathcal{L} \to \widetilde{Q} \tag{5.3.3}$$

defined as follows: consider the matching maps $m_q : (L_q)_i \to (\lambda_q)_i$. Then $\mathcal{L} = \sqcup_{q \in Q, i} (L_q)_i$ and $m|_{L_q} = m_q$.

We write **Higgs** for the category of rank n Higgs bundles (E, ϕ) with ϕ generic. Let **Spec** be the category whose objects are pairs $(\pi, \mathcal{L})$ where $\pi : \widetilde{Q} \to Q$ is an unramified n-sheeted covering map, and $\mathcal{L} \to \widetilde{Q}$ is a complex line bundle.

Theorem 5.3.1. Spectral Correspondence - Classical Version*: There is an equivalence of categories:*

$$\begin{array}{ccc} \textbf{\textit{Higgs}} & \longrightarrow & \textbf{\textit{Spec}} \\ (E, \phi) & \longmapsto & (\widetilde{Q}, \mathcal{L}) \end{array} \tag{5.3.4}$$

Proof. The above discussion explains how to obtain spectral data from Higgs data. Conversely, given $(\widetilde{Q}, \mathcal{L})$, Higgs data is obtained by $E = \pi_* \mathcal{L}$ and $\phi = \pi_* \tau$, where the tautological section $\tau : \widetilde{Q} \to End(\mathcal{L})$ is defined as $\tau(q, \lambda) = \lambda \mathbf{1}_{\mathcal{L}}$.

We omit the proof of the equivalence for morphisms, as it is not essential for our purposes. □

Remark 5.3.2. If ϕ is regular, then the pushforward of the spectral line bundle by m will not recover E: one needs to pushforward the whole generalized eigenspace associated to an eigenline; i.e., one needs to consider a more general sheaf $\mathcal{L}' \to \widetilde{Q}$ such that on the locus $\Delta_\phi \subseteq Q$ where ϕ_q is non-generic, $\mathcal{L}'_{q,\lambda_q}$ jumps in rank and is given by the λ_q-Jordan block of ϕ_q. Such a locus is codimension two in Q. In particular, when Q is a 3-manifold, Δ_ϕ is a graph in Q. We will have more to say about this further on.

Theorem 5.3.1 is the most raw form of the spectral correspondence. One can also consider more general notions of Higgs bundles: one can "twist" ϕ by requiring its coefficients to be valued in a sheaf of commutative groups $\mathcal{F}$ over Q, and also require ϕ to satisfy some constraint (e.g., being compatible with fixed geometric structures on Q or E). In this situation, the spectral data must be suplemented with additional structure in order to reconstruct (E, ϕ). We will be interested in the case $\mathcal{F} = \Omega^1_Q$ with constraints given by the axioms 5.1.2 for a flat Higgs bundle.

Definition 5.3.3. Let (E, h, ∇, θ) be a flat Higgs bundle over Q. The *Spectral Cover* associated to θ is the subvariety $S_\theta \subset T^*Q$ defined by:

$$S_\theta = \Big\{(q, s_q); \det(s_q \otimes \mathbf{1}_{E_q} - \theta_q) = 0\Big\} \tag{5.3.5}$$

Definition 5.3.4. *Flat Spectral Data - unramified case:* Let (E, h, A, θ) be a rank n flat Higgs bundle over a compact flat 3-manifold (Q, δ). Assume θ is generic. We define *flat spectral data* to be:

1. A n-sheeted, unramified covering map $\pi : S_\theta \to Q$ given by the characteristic polynomial of θ.

2. A line bundle $\mathcal{L} \to S_\theta$ determined by the eigenlines of θ_q

3. A hermitian metric $\widetilde{h}$ on $\mathcal{L}$ determined by h

4. A hermitian flat connection $\widetilde{A}$ on $\mathcal{L}$ determined by A

5. A Lagrangian embedding $\ell : S_\theta \to T^*Q$ satisfying $Im(d\ell) \subset H_\delta$.

Remark 5.3.5. The last condition admits a second interpretation: we view ℓ as a Lagrangian section of the pull-back bundle $\pi^*T^*Q \to S_\theta$ and take its covariant derivative with respect to the pullback $\pi^*\delta$ of the flat connection δ on $T^*Q \to Q$. Then the condition is that $\nabla_{\pi^*\delta}\ell = 0$. If one is interested in non-flat Q, the condition $Im(d\ell) \subset H_\delta$ is simply dropped. As we will see below, the flat spectral correspondence works for any compact Riemannian manifold Q.

Definition 5.3.6. *Flat Spectral Data - totally ramified case:* With the same notation as before, suppose θ is central - i.e., diagonalizable with all eigenvalues equal. Then its flat spectral data is as before, except that $\mathcal{L}$ is replaced by a rank n complex vector bundle $\mathcal{E} \to S_\theta$. Moreover, note that π is now totally ramified.

We now come to the main result of this section. Let **FlatHiggs** be the set of flat Higgs bundles (E, h, A, θ) over a compact Riemannian Q and **FlatSpec** the set of flat spectral data $(\pi, \mathcal{L}, \widetilde{h}, \widetilde{A}, \ell)$ on Q.

Theorem 5.3.7. (Spectral Correspondence for flat Higgs bundles [PW11]) *There is an equivalence:*

$$\textbf{\textit{FlatHiggs}} \longleftrightarrow \textbf{\textit{FlatSpec}} \tag{5.3.6}$$

where flat Higgs bundles are taken to be either unramified or totally ramified, and the spectral data is chosen appropriately for each case.

Proof. Given a flat Higgs bundle (E, h, A, θ), we already know how to construct $\pi : S_\theta \to Q$, $\ell : S_\theta \to T^*Q$ and $\mathcal{L} \to S_\theta$. The metric and flat connection are just given by pullback: $\widetilde{h} := \pi^* h$ and $\widetilde{A} := \pi^* A$, so the compatibility condition $\nabla_{\tilde{A}} \widetilde{h} = 0$ is preserved.

Now, use the hamonicity condition on $h : \widetilde{Q} \to G/K$ to identify $\theta = dh$. The condition $\nabla_A \theta = d\theta + A \wedge \theta = 0$ can be written as equations for the r components of θ under the identification, where $r = \operatorname{rank}(G)$. Since $\nabla_A^2 = 0$, we can locally gauge away A, so that that the equations become $d\theta = 0$. Let (x_i, y_i) be coordinates in T^*Q such that $\omega = \sum dy_i \wedge dx_i$ and dualize $\theta = \sum \theta_i(x)dx_i$ via the semi-flat metric on T^*Q to obtain $\theta' = \sum \theta_i(x)dy_i$. The embedding given by $\ell(q, s_q) = \det(s_q \otimes \mathbf{1}_E - \theta_q)$ is Lagrangian if and only if:

$$\omega|_{\ell(Q)} = \sum_{i=1}^{3} d\theta' \wedge dx_i = \sum_{i=1}^{3} d(\theta_i dx_i) = d\theta = 0 \tag{5.3.7}$$

Conversely, given spectral data $(\pi, \mathcal{L}, \widetilde{h}, \widetilde{A}, \ell)$, one constructs E and θ as usual, and $h = \pi_* \widetilde{h}$, $A = \pi_* \widetilde{A}$ are well-defined because π is a local isometry. The spectral data also guarantees that the components of θ are simultaneously diagonalizable, hence $\theta \wedge \theta = 0$. The conditions $\nabla_A^2 = 0$ and $\nabla_A h = 0$ follow from the same conditions for $(\widetilde{h}, \widetilde{A})$. Finally, the condition $\nabla_A \theta = 0$ is obtained simply by reversing the above argument for the section ℓ to be Lagrangian.

$\square$

Definition 5.3.8. The *Hitchin map* is defined by:

$$\mathfrak{H} : \mathbf{FlatHiggs} \to \bigoplus_{i=1}^{n} \Gamma(Q, (T_\delta^* Q)^{\otimes i})$$

$$(E, h, A, \theta) \mapsto (p_1(\theta), \ldots, p_n(\theta))$$

We will make a careful study of the properties of this map in future work. For now, we observe two differences from the holomorphic category: one, $\mathfrak{H}$ is not surjective in general; and two, even when restricted to its image, $\mathfrak{H}$ will not define an integrable system structure, at least not in the usual sense. The reason is, character varieties of compact three-manifolds are in general not symplectic; however, they admit canonical (-1)-*shifted symplectic structures* [PTVV11], so they are symplectic in this derived sense. Thus, assuming a non-abelian Hodge theorem holds in the flat setup, one would expect that the moduli space of flat Higgs bundles inherits the (-1)-shifted symplectic structure, and hope that $\mathfrak{H}$ defines a notion of integrable system compatible with this structure.

In the next chapter, we will study the Hitchin map $\mathfrak{H}$ in a specific example related to the mirror of the Hatzsche-Wendt Calabi-Yau X. In that example, we will see that $\mathfrak{H}$ is essentially a smooth model of the SYZ fibration mirror to $f : X \to \mathbb{R}^3_{\mathbf{Y}}$.

Chapter 6

SYZ Mirrors

6.1 The Hantzsche-Wendt mirror

Recall from Chapter 4 that the manifold $X := T^*\mathcal{G}_6$ admits a torus fibration:

$$f : X \to \mathbb{R}^3_{\mathbf{Y}} \tag{6.1.1}$$

where $\mathbb{R}^3_{\mathbf{Y}} := \mathbb{R}^3/\mathbb{K}$ is an orbifold of $\mathbb{R}^3$ with singular locus along the three coordinate axes. Recall also that Theorem 4.3.2 implies that $X \to \mathbb{R}^3_{\mathbf{Y}}$ admits a semi-flat Calabi-Yau metric, and hence the smooth fibers are special Lagrangian tori.

Restricting f to the smooth locus $\mathcal{S} := \mathbb{R}^3_{\mathbf{Y}} \backslash \mathbf{Y}$, the *SYZ Mirror Symmetry Conjecture* [SYZ96] predicts that the mirror Calabi-Yau space of $X \backslash f^{-1}(\mathbf{Y})$ is the total space $\widehat{X}$ of the dual torus fibration over $\mathcal{S}$, with its semi-flat metric. To include singular fibers in the conjecture, one needs to modify the complex structure on $\widehat{X}$

by "instanton corrections" coming from holomorphic disks bounded by the singular fibers of X. The precise way in which the corrections must be included is one of the delicate points in Mirror Symmetry.

For some special types of singular fibrations, one can construct the mirror rather effortlessly; e.g., if a $\mathbf{T}$-fibration is $\mathbb{R}$-*simple* in the sense of Gross [Gro98] [Gro99], then one can prove that the dual fibration has mirror Hodge numbers. Unfortunately, a necessary condition for a fibration to be $\mathbb{R}$-simple is that the fibers must be irreducible, which is not the case for the singular fibers of $X \to \mathbb{R}^3_{\mathbf{Y}}$.

However, the smooth special Lagrangian fibration[1] $d : T^*\mathbf{T} \to \mathbb{R}^3$ *is* $\mathbb{R}$-simple. Lemma 4.3.5 identifies the SYZ fibration for the Hantzsche-Wendt Calabi-Yau X as a $\mathbb{K}$-quotient of d. We propose that the mirror $\widehat{X}$ should be obtained as a suitable $\mathbb{K}$-quotient of the dual fibration $\hat{d}$. This dual fibration is given by:

$$\widehat{d} : (\mathbb{C}^*)^3 \to \mathbb{R}^3 \tag{6.1.2}$$

using again the identification of a fixed fiber of $(\mathbb{C}^*)^3 \cong T\mathbf{T}$ with $\mathbb{R}^3$. The fibers are given by dual tori $\widehat{\mathbf{T}}$, which parametrize $U(1)$ local systems on $\mathbf{T}$. Hence the proposed $\mathbb{K}$-action on $(\mathbb{C}^*)^3$ is given by the usual action 2.1.3 on $\mathbb{R}^3$ and pullback of local systems on $\widehat{\mathbf{T}}$. Introducing coordinates z_1, z_2, z_3 on $(\mathbb{C}^*)^3$, the action is described by:

[1]Here for simplicity we identify a fixed fiber of $T^*\mathbf{T} \to \mathbf{T}$ with $\mathbb{R}^3$ and use the flat structure to identify all other fibers with the fixed one.

$$\alpha(z_1, z_2, z_3) = (z_1, z_2^{-1}, z_3^{-1})$$

$$\beta(z_1, z_2, z_3) = (z_1^{-1}, z_2, z_3^{-1})$$

where $\mathbb{K} = \langle \alpha, \beta \rangle$.

Therefore, our proposed mirror is the complex smooth orbifold:

$$\widehat{X} := \left[(\mathbb{C}^*)^3 \big/ \mathbb{K} \right] \tag{6.1.3}$$

with a special Lagrangian torus fibration:

$$\widehat{f} : \widehat{X} \to \mathbb{R}^3_{\mathbf{Y}} \tag{6.1.4}$$

In the next section, we explain how SYZ transforms A-branes into B-branes.

6.2 Mirror B-branes

In previous chapters we studied the moduli space of complexified G_2-structures $\mathcal{M}^{\mathbb{C}}_{G_2}$ on a Hantzsche-Wendt G_2-platyfold of type A_n. We identified it with the moduli space of n A-branes "wrapped" on the zero-section $\mathcal{G}_6$ of the flat Calabi-Yau space $T^*\mathcal{G}_6 \to \mathcal{G}_6$. We also proved this same space has a different Calabi-Yau structure, where $\mathcal{G}_6$ appears as the (multiplicity four) central fiber of a special Lagrangian torus fibration $f : T^*\mathcal{G}_6 \to \mathbb{R}^3_{\mathbf{Y}}$. One can think of this second structure as a "large volume limit", which for our purposes means that it represents a point in the moduli space of Calabi-Yau structures on $X := T^*\mathcal{G}_6$ where Mirror Symmetry applies.

We are interested in understanding how Mirror Symmetry acts on A-branes. The A-brane structure on a smooth torus fiber T_b is given by:

- An embedding as a special Lagrangian fiber $\mathbf{T}_b \hookrightarrow X$
- A hermitian line bundle $\mathcal{L} \to \mathbf{T}_b$
- A flat unitary connection A on $\mathcal{L}$

Intuitively, the SYZ mirror transformation maps the data $(T_b, \mathcal{L}, A)$ simply to the local system $(\mathcal{L}, A)$, seen as a point $p_{\mathbf{T}}$ in the mirror torus fiber $\hat{T}$. More precisely, the mirror map is given by the *Fourier-Mukai functor* $\mathbb{FM}$ of Arinkin-Polishchuk [AP98] and Leung, Yau and Zaslow [LYZ02], and the image is a *skyscraper sheaf* $\mathcal{F}_{p_{\mathbf{T}}}$ at the point. This is a coherent sheaf on $\widehat{X}$, hence an element of the category of B-branes $D^b(\hat{X})$.

Our considerations in Chapter 4 led us to define the data of "n A-branes wrapped on $\mathbf{T}_b$" as:

- An embedding as a special Lagrangian fiber $\mathbf{T}_b \hookrightarrow X$
- A rank n complex vector bundle $\mathcal{E} \to \mathbf{T}_b$
- A flat complex connection A on $\mathcal{E}$

It is then natural to extend the SYZ mirror map to map "n A-branes supported at $\mathbf{T}_b$" to "n B-branes supported at $\widehat{\mathbf{T}}_b$". That is, the n B-branes are described by a (possibly unreduced) scheme $\sum_{i=1}^{n} p_i$ supported at $\widehat{\mathbf{T}}_b$, or more precisely, by the direct sum sheaf $\bigoplus_{i=1}^{n} \mathcal{F}_{p_i}$. In terms of the Fourier-Mukai functor, this says that $\mathbb{FM}$ maps complex flat connections on $\mathbf{T}_b \subset X$ to length n subschemes supported at $\widehat{\mathbf{T}}_b \subset \hat{X}$.

Since the central fiber of f is not singular in the usual sense (it is a reducible scheme consisting of four coincident copies of $\mathcal{G}_6$, each of which is smooth), we assume the same result holds. Then mirror symmetry for branes predicts an identification:

$$\mathrm{Char}(\pi, SL(n, \mathbb{C})) \cong \mathrm{Hilb}^n_{\mathbf{Y}}(\widehat{X}) \tag{6.2.1}$$

where $\mathrm{Hilb}^n_{\mathbf{Y}}(\widehat{X})$ is the *Hilbert scheme of n points supported at the fiber over the vertex of* $\mathbf{Y}$. It parametrizes length n subschemes of $\widehat{X}$ supported at that fiber, so it is exactly the moduli space of n B-branes we seek to describe.

Since $\widehat{X}$ is an orbifold, our goal will be to build a smooth auxiliary space whose Hilbert scheme coincides with that of $\widehat{X}$. This is accomplished by choosing an appropriate crepant resolution of $\widehat{X}$. Note that, since our moduli space parametrizes *stable* B-branes[2], different crepant resolutions may give different answers.

[2]The stability condition on the A-side is the requirement that the monodromy of the local system is irreducible.

6.2.1 The Bridgeland-King-Reid crepant resolution

Let Y be an algebraic space where a finite group G of automorphisms acts. Consider the orbit space Y/G. The *space of G-clusters* $\widetilde{Y} := G-\mathrm{Hilb}(Y)$ is defined as follows: consider the induced action of G on $\mathrm{Hilb}^{|G|}(Y)$, and let $\mathrm{Fix}(G)$ be the subscheme of fixed points. Then $\widetilde{Y}$ is the closure of the set of free orbits in $\mathrm{Fix}(G)$. For this reason, $\widetilde{Y}$ is also known in the literature as the Hilbert scheme of G-orbits. A more thorough discussion of this space can be found in [Blu11].

The derived McKay correspondence says that $\widetilde{Y}$ is a crepant resolution of $[Y/G]$, and moreover, there is an equivalence of categories [Bri07]:

$$\mathcal{D}(\widetilde{Y}) \cong \mathcal{D}^G(Y) \tag{6.2.2}$$

between the derived category of coherent sheaves on $\widetilde{Y}$ and the G-equivariant coherent sheaves on Y. In particular, since by definition $\mathcal{D}([Y/G]) := \mathcal{D}^G(Y)$ - where $[Y/G]$ is the *stack* of G-orbits - the equivalence, when restricted to sheaves generated by skyscraper sheaves implies:

$$\mathrm{Hilb}^n(\widetilde{Y}) \cong \mathrm{Hilb}^n([Y/G]) \tag{6.2.3}$$

Thus $\widetilde{Y} := \mathbb{K}-\mathrm{Hilb}((\mathbb{C}^*)^3)$ is a rather natural candidate for a “geometric” model of $X^\vee = [(\mathbb{C}^*)^3/\mathbb{K}]$. However, one can readily see that this crepant resolution is not the right one: in fact, $\mathrm{Hilb}^n_{\mathbf{Y}}(\widetilde{Y})$ does not match the character variety of $\mathcal{G}_6$ even for low n.

At this point, there are two possible routes. The most natural one would be to choose a Bridgeland stability condition and restrict the Hilbert scheme to contain only stable coherent sheaves. The second route, which is the one we will pursue here, is to produce a different crepant resolution. Although more convoluted, the reason we choose this second approach is that it not only gives the correct answer (at least for $n = 2$), but it will also elucidate how the Fourier-Mukai transform connects with our previous discussion of flat Higgs bundles and spectral covers.

6.3 The spectral mirror

We now introduce a "crepant resolution" of $\widehat{X}$ and show it is a good geometric model for the mirror, in the sense that its moduli space of two B-branes matches $\mathcal{M}_A(X) = \mathrm{Char}(\mathcal{G}_6, SL(2,\mathbb{C}))$. We conjecture that the result holds for all n.

The reason we use

To construct this space we use a combination of SYZ and the flat spectral construction. For this reason we call it the *spectral mirror*, and denote it $\check{X}$. The purpose of this section is twofold: first, we motivate the following definition:

Definition 6.3.1. Let $\mathcal{M}_{\text{Higgs}}^{SO(4,\mathbb{C})}$ denote the moduli space of flat $SO(4,\mathbb{C})$-Higgs bundles on $\mathcal{G}_6$. The *spectral mirror* of X is the subspace $\check{X} := \mathcal{M}_{\text{Higgs}}^{\mathbb{K}}$ of $\mathcal{M}_{\text{Higgs}}^{SO(4,\mathbb{C})}$ consisting of Higgs bundles whose spectral cover has Galois group isomorphic to $\mathbb{K}$.

In order for this to be useful, we need to prove that $\check{X}$ admits a SYZ fibration

$\check{f} : \check{X} \to \mathbb{R}^3_{\mathbf{Y}}$ that agrees with $\widehat{f}$ outside the discriminant locus $\mathbf{Y}$. This is the content of Proposition 6.3.3 below.

We then prove the following result:

Theorem 6.3.2. *There is a map:*

$$Char(\pi, SL(2,\mathbb{C})) \to PHilb^2_{\mathbf{Y}}(\check{X}) \tag{6.3.1}$$

which is a homeomorphism when restricted to

$$Char^0(\pi, SL(2,\mathbb{C})) = Char(\pi, SL(2,\mathbb{C}))\backslash \{p_1, p_2, p_3\}$$

It follows that the moduli space of two coincident A-branes on X is (almost) homeomorphic to the moduli space of two coincident B-branes on $\widehat{X}$.

Given the special Lagrangian fibration $f : X \to \mathbb{R}^3_{\mathbf{Y}}$ and a smooth fiber $\mathbf{T}_b$, SYZ predicts that the smooth fibers $\widehat{\mathbf{T}}_b$ of $\widehat{f} : \widehat{X} \to \mathbb{R}^3_{\mathbf{Y}}$ (i.e., $b \in \mathbb{R}^3_{\mathbf{Y}}\backslash\mathbf{Y}$) parametrize $U(1)$-local systems on $\mathbf{T}_b$. Thus, we write the points of $\widehat{\mathbf{T}}_b$ as (L_b, a_b).

Now assume (L_b, a_b) is a deformation of a local system (L_0, a_0) on one of the irreducible components of $f^{-1}(0)$ i.e., a copy of $\mathcal{G}_6$. I.e., we imagine that if we connect b to 0 by a smooth path not crossing $\mathbf{Y}$, each fiber $f^{-1}(b')$ is endowed with a local system $(L_{b'}, a_{b'})$ such that $(L_{b'}, a_{b'}) \rightsquigarrow (L_0, a_0)$ as $b' \to 0$. So we are given:

1. A four-sheeted unramified normal covering $g : \mathbf{T}_b \to \mathcal{G}_6$ with Galois group $\mathrm{Gal}(\mathbf{T}_b/\mathcal{G}_6) \cong \mathbb{K}$

2. A line bundle $L_b \to \mathbf{T}_b$

3. A flat $U(1)$-connection a_b on L_b

4. A Lagrangian embedding $\mathbf{T}_b \hookrightarrow X \cong T^*\mathcal{G}_6$ given by the inclusion as a SYZ fiber.

In other words, a local system on a smooth fiber is exactly the data of a flat spectral cover of $\mathcal{G}_6$ without the harmonicity condition. By the flat spectral correspondence, this is equivalent to the Higgs data on $\mathcal{G}_6$; i.e., we are given:

1. A rank 4 complex vector bundle $F \to \mathcal{G}_6$

2. A flat unitary connection A on F

3. A flat Higgs bundle $\theta \in \Omega^1(\mathcal{G}_6, End(F))$

Moreover the condition that $\mathrm{Gal}(\mathbf{T}_b/\mathcal{G}_6) \cong \mathbb{K}$ imposes a further restriction on θ, which we now explain.

Let $p : T^*\mathcal{G}_6 \to \mathcal{G}_6$ be the cotangent projection. Recall that in the proof of 5.3.1, $\theta = g_*(\tau|_{\mathbf{T}_b})$, where $\tau : T^*_\delta\mathcal{G}_6 \to p^*\Omega^1_\delta(\mathcal{G}_6)$ is the tautological section. The flat spectral cover is:

$$\mathbf{T}_b = \{\det(\tau\mathbf{1} - p^*\theta) = 0\} \subset p^*\Omega^1_\delta(\mathcal{G}_6) \tag{6.3.2}$$

From now on we drop the pullback by p from our notation in order to make it less cumbersome.

The sheets of the flat spectral cover have symmetry $\mathbb{K}$, the dihedral group with four elements. Note that $\mathbb{K}$ is the Weyl group of $SO(4)$, so our θ takes values in

$\mathfrak{so}(4, \mathbb{C})$. We can prove this explicitly: since the spectral cover has Galois group $\mathbb{K}$, it follows that at each q, the eigenvalues of θ_q are arranged in the shape of a square in $\mathbb{C}$ (i.e., a regular 4-gon). We write them as:

$$\mathrm{Eig}_\theta = \{\lambda, i\lambda, -\lambda, -i\lambda\} \subset \Omega^1_\delta(\mathcal{G}_6) \otimes \mathbb{C} \tag{6.3.3}$$

Since the cover is unramified, one necessarily has $\lambda \neq 0$. Moreover, without loss of generality we can take $\lambda \in \mathbb{R}$.[3]

We have from 6.3.3 that $Tr(\theta) = 0$, so it is not surprising that θ preserves a metric h on E. With respect to a local frame that diagonalizes θ, it takes the form:

$$h = \begin{pmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \tag{6.3.4}$$

for non-vanishing functions $A, B : \mathcal{G}_6 \to \mathbb{C}$. Thus $\theta \in \Omega^1_\delta(\mathcal{G}_6, \mathfrak{so}_\mathbb{C}(h))$.

It is also clear from 6.3.3 that $Tr(\theta^2) = 0$: if we expand the formula for the characteristic polynomial, we have:

[3] Although it is possible that all eigenvalues have non-zero imaginary part, the choice of complexification of the sheaf $\Omega^1_\delta(\mathcal{G}_6)$ is immaterial; we could have chosen one such that $\lambda \in \mathbb{R}$, or, as we did here, we can pick the global one $\Omega^1_\delta(\mathcal{G}_6) \otimes \mathbb{C}$ and choose a section of real structures such that $\lambda \in \mathbb{R}$.

$$\det(\tau - \theta) = \tau^4 + \det(\theta) \tag{6.3.5}$$

In general, a basis for the invariant polynomials on $\mathfrak{so}(2n, \mathbb{C})$ is given by combinations of powers of Tr and the *Pffafian* $p_n = \sqrt{\det} \in \mathrm{Sym}^n(\mathfrak{so}(2n, \mathbb{C}))$. In our situation, only $p_2(\theta)$ is non-vanishing.

Let $\mathcal{M}_{\mathrm{Higgs}}^{SO(4,\mathbb{C})}$ be the *moduli space of flat $SO(4, \mathbb{C})$-Higgs bundles over $\mathcal{G}_6$*, and let $\mathcal{M}_{\mathrm{Higgs}}^{\mathbb{K}} \subset \mathcal{M}_{\mathrm{Higgs}}^{SO(4,\mathbb{C})}$ be the subspace consisting of flat Higgs bundles whose spectral covers have Galois group $\mathbb{K}$. The above discussion implies that the Hitchin map restricts to:

$$\mathscr{H} : \mathcal{M}_{\mathrm{Higgs}}^{\mathbb{K}} \to H^0(\mathcal{G}_6, \mathrm{Sym}^2_\delta(T^*\mathcal{G}_6))$$

$$(E, h, A, \theta) \to p_2(\theta)$$

and note that the base is locally isomorphic to $\mathbb{R}^6$.

We have shown that $\mathcal{M}_{\mathrm{Higgs}}^{\mathbb{K}}$ solves the deformation problem, but to show it is the correct geometric model for the mirror we need to endow it with the structure of a SYZ fibration mirror to $\pi : X \to \mathbb{R}^3_{\mathbf{Y}}$. Let $\mathcal{B} = Im(\mathscr{H})$.

Proposition 6.3.3. *The Hitchin map $\mathscr{H} : \mathcal{M}_{Higgs}^{\mathbb{K}} \to \mathcal{B}$ agrees with $\widehat{f} : \widehat{X} \to \mathbb{R}^3_{\mathbf{Y}}$ outside the discriminant locus $\mathbf{Y}$.*

Proof. We need to show two things: that $\mathcal{B} \cong \mathbb{R}^3_{\mathbf{Y}}$ and that $\mathscr{H}^{-1}(b) \cong \widehat{\mathbf{T}}_b$ for $b \in \mathcal{B} \backslash \mathbf{Y}$.

We start with a local discussion. See $\mathrm{Sym}^2(\mathbb{R}^3) \cong \mathbb{R}^6$ as the symmetric matrices in $End_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^3) \cong \mathbb{R}^9$. The image of $\mathscr{H}$ cuts out an algebraic subspace of $\mathbb{R}^6$. Suppose $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Then the expression for $\mathcal{B}$ is:

$$p_2(\theta) = \begin{pmatrix} \lambda_1^2 dx \otimes dx & \lambda_1\lambda_2 dx \otimes dy & \lambda_1\lambda_3 dx \otimes dz \\ \lambda_1\lambda_2 dx \otimes dy & \lambda_2^2 dy \otimes dy & \lambda_2\lambda_3 dy \otimes dz \\ \lambda_1\lambda_3 dx \otimes dz & \lambda_2\lambda_3 dy \otimes dz & \lambda_3^2 dz \otimes dz \end{pmatrix} \tag{6.3.6}$$

which is of dimension 3.

Changing coordinates ($w_j = \lambda_j^2$, $u_j = \lambda_{j+1}\lambda_{j+2}$) we see that locally, $Im(\mathscr{H})$ is the intersection of three quadrics in $\mathbb{R}^6$:

$$Im(\mathscr{H}) = \begin{cases} u_1^2 = w_2 w_3 \\ u_2^2 = w_1 w_3 \\ u_3^2 = w_1 w_2 \end{cases} \tag{6.3.7}$$

It is clear that by adding the three equations in 6.3.7 and defining $t = w_1 w_2 + w_1 w_3 + w_2 w_3 \geqslant 0$ we get:

$$u_1^2 + u_2^2 + u_3^2 = t \tag{6.3.8}$$

=a cone over $\mathbf{S}^2$, which is topologically $\mathbb{R}^3$. Conversely, start from 6.3.8 and define t_1, t_2, t_3 by:

$$\begin{cases} t_2/t_1 = u_1^2/u_2^2 := s \\ t_3/t_1 = u_1^2/u_3^2 := v \end{cases} \tag{6.3.9}$$

Then:

$$\begin{cases} t_2 = st_1 \\ t_3 = vt_1 \\ t_1t_2 + t_1t_3 + t_2t_3 = t \end{cases} \tag{6.3.10}$$

from which it follows that $(s+v+sv)t_1^2 = t$, and only the positive t_1 is a solution. Hence, we get a unique triple (t_1, t_2, t_3).

Now, to get a global description of $Im(\mathscr{H})$, we need to include the action of $\mathbb{K}$ on $\mathrm{Sym}^2_\delta(T^*\mathcal{G}_6)$. The group $\mathbb{K}$ acts at each cross-section of the cone 6.3.8 by switching two elements in $\{w_1, w_2, w_3\}$, and the action has stabilizers precisely on the three coordinate axes of $\mathbb{R}^3_{(u_1^2,u_2^2,u_3^2)}$. Hence, $Im(\mathscr{H})$ is a cone over the sphere with three $\mathbb{Z}_2$-orbifold points $\mathbf{S}^2_{(u_1,u_2,u_3)}$, i.e.:

$$Im(\mathscr{H}) \cong \mathbb{R}^3_{\mathbf{Y}} \tag{6.3.11}$$

is the $\mathbf{Y}$-vertex, as expected. Note that the tip of the cone is taken to be a fixed point of the action, i.e. it is a $\mathbb{K}$-orbifold point.

Now consider the fibers $\mathscr{H}^{-1}(b)$. By definition, the spectral cover $\mathbf{T}_b$ only depends on a point b in the base of the Hitchin map. By the flat spectral correspondence, the fiber $\mathscr{H}^{-1}(b)$ parametrizes the remaining flat spectral data $(L, \widetilde{h}, \widetilde{A})$, i.e., $U(1)$-local systems on $\mathbf{T}_b$. Therefore, $\mathscr{H}^{-1}(b) \cong \widehat{\mathbf{T}}_b$.

$\square$

This finishes the discussion motivating Definition 6.3.1. We now proceed to prove Theorem 6.3.2.

We use Theorem 5.1.8 to translate the computation of $\mathcal{M}_{\text{Higgs}}^{SO(4,\mathbb{C})}$ to finding the moduli of flat $SO(4,\mathbb{C})$-connections. I.e., we need to compute $\mathrm{Char}^0(\mathcal{G}_6, SO(4,\mathbb{C}))$.

One way to do it is to repeat the method of section 5.2 and find its image in:

$$\mathrm{Char}^0(\mathbb{Z}^3, SO(4,\mathbb{C})) \cong {}^{(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2}\big/_{\mathbb{Z}_2 \times \mathbb{Z}_2} \tag{6.3.12}$$

We emphasize that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ in this formula has nothing to do with $\mathbb{K}$; it is just the Weyl group of $SO(4,\mathbb{C})$. The action is diagonal and one generator permutes the two $\mathbb{C}^*$'s, while the other inverts them simultaneously. So $\mathrm{Char}^0(\mathbb{Z}^3, SO(4,\mathbb{C}))$ looks like $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{C}^6$.

There is also a $\mathbb{K}$-action on this space induced from the monodromy representation. It acts as follows: write each $\mathbb{C}^2 = \mathbb{C}_{i,1} \times \mathbb{C}_{i,2}$ for $i = 1,2,3$. Then for fixed $j \in \{1,2\}$, $\mathbb{K}$ acts on $\mathbb{C}_{1,j} \times \mathbb{C}_{2,j} \times \mathbb{C}_{3,j}$ by the usual monodromy action. Moreover, $\mathrm{Char}^0(\mathcal{G}_6, SO(4,\mathbb{C}))$ lies in $\mathrm{Fix}(\mathbb{K})$, which is just $\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}}$. One can check exactly as we did in section 5.2 that the image covers the whole $\mathrm{Fix}(\mathbb{K})$, so:

$$\mathrm{Char}^0(\mathcal{G}_6, SO(4,\mathbb{C})) \cong \mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}} \tag{6.3.13}$$

In order to compute the Hilbert scheme $\mathrm{PHilb}^2_{\mathbf{Y}}(\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}})$, we need to look at the subset of $\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}}$ whose points are fixed by the action of the permutation group $\Sigma_2 \cong \mathbb{Z}_2$. This is just given by the diagonal $\mathbf{Y}_{\mathbb{C}} \subset \mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}}$. When we quotient by permutations, we get a space $[\mathbf{Y}_{\mathbb{C}}/\Sigma_2]$ which is a copy of $\mathbf{Y}_{\mathbb{C}}$ consisting purely of Σ_2-orbifold points. The restriction of the Hilbert-Chow morphism gives:

$$\mathrm{PHilb}^2(\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}})|_{\mathbf{Y}_{\mathbb{C}}} = \mathrm{Hilb}^2(\mathbf{Y}_{\mathbb{C}}) \to [\mathbf{Y}_{\mathbb{C}}/\Sigma_2] \tag{6.3.14}$$

a resolution of singularities, which in this case consists simply of two coincident copies of $\mathbf{Y}_{\mathbb{C}}$.

Topologically, $\mathrm{Hilb}^2(\mathbf{Y}_{\mathbb{C}})$ looks like a $\mathbb{P}^1$-bundle over $\mathbf{Y}_{\mathbb{C}} \backslash \{0\}$ with a trident of $\mathbb{P}^1$'s over 0, which we denote $\mathbf{Y}_{\mathbb{P}^1}$ for obvious reasons. Hence:

$$\mathrm{PHilb}^2_{\mathbf{Y}}(\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}}) \overset{\text{homeo.}}{\cong} \mathbf{Y}_{\mathbb{P}^1} = \mathbb{P}\mathrm{Char}^0(\mathcal{G}_6, SL(2,\mathbb{C})) \tag{6.3.15}$$

and this establishes Theorem 6.3.2.

Thus, we have proved that *the moduli space of two D_6 A-branes wrapping $\mathcal{G}_6$ in X is homeomorphic to the moduli space of two coincident D_3 B-branes on $\widehat{X}$.* These spaces are topologically $\mathbf{Y}_{\mathbb{C}}$, which is a complexified version of the space of singular orbits of the moduli space of two $SU(2)$ Atiyah-Hitchin monopoles. This is as to be expected, as our D-branes are monopoles with gauge group $SL(2,\mathbb{C})$.

6.3.1 A different method to compute $\mathrm{Char}^0(\mathcal{G}_6, SO(4,\mathbb{C}))$

To end this section, we sketch a different argument to compute $\mathrm{Char}^0(\mathcal{G}_6, SO(4,\mathbb{C}))$. Recall that $Spin(4) \cong SU(2) \times SU(2)$, and $SU(2) \cong Im(\mathbb{H})$. Under these isomorphisms, the two-fold universal covering homomorphism $Spin(4) \to SO(4)$ is interpreted as multiplication of imaginary quaternions. After complexifying, the "quaternionic version" of the complex universal covering homomorphism $Spin(4,\mathbb{C}) \to SO(4,\mathbb{C})$ is $\kappa : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C})$.

Here is an explicit description of κ: let $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ act on $\mathbb{C}^2 \otimes \mathbb{C}^2$ by matrix multiplication. This action preserves the following pairing:

$$\langle u \otimes v, w \otimes t \rangle = \det(u \otimes w^*) \det(v \otimes t^*) \tag{6.3.16}$$

The pairing is symmetric and non-degenerate, hence the desired map is

$$\kappa : \left(\begin{pmatrix} a_1 & \overline{a_2} \\ -a_2 & \overline{a_1} \end{pmatrix}, \begin{pmatrix} a_3 & \overline{a_4} \\ -a_4 & \overline{a_3} \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 & \overline{a_2} & 0 & 0 \\ -a_2 & \overline{a_1} & 0 & 0 \\ 0 & 0 & a_3 & \overline{a_4} \\ 0 & 0 & -a_4 & \overline{a_3} \end{pmatrix} \tag{6.3.17}$$

and it satisfies $Im(\kappa) \subset SO(\langle -, - \rangle)$.

Consider the exact sequence:

$$1 \to \mathbb{Z}_2 \to SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \xrightarrow{\kappa} SO(4,\mathbb{C}) \to 1 \tag{6.3.18}$$

where $\mathbb{Z}_2 \hookrightarrow SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ is the diagonal embedding. This induces:

$$1 \to Hom(\pi, \mathbb{Z}_2) \to \mathrm{Hom}(\pi, SL(2,\mathbb{C}))^2 \xrightarrow{\overline{\kappa}} \mathrm{Hom}(\pi, SO(4,\mathbb{C})) \tag{6.3.19}$$

where we recall that $\pi := \pi_1(\mathcal{G}_6)$. Now, unless we know the map $\overline{\kappa}$ precisely, there is not much else that can be done. For the sake of argument, assume for a moment that it is surjective (we will explain in a moment the meaning of this condition). If this is the case, then combined with the fact that $\mathrm{Hom}(\pi, \mathbb{Z}_2) \cong \mathbb{K}$, we have:

$$\mathrm{Hom}(\pi, SO(4,\mathbb{C})) \cong \mathrm{Hom}(\pi, SL(2,\mathbb{C}))^2/\mathbb{K} \tag{6.3.20}$$

so the character variety is essentially:

$$\mathrm{Char}(\pi, SO(4,\mathbb{C})) \cong \mathrm{Hom}(\pi, SL(2,\mathbb{C})^2) \Big/ \mathbb{K} \rtimes SO(4,\mathbb{C}) \tag{6.3.21}$$

i.e., it is a quotient of $\mathrm{Char}(\pi, SL(2,\mathbb{C})^2)$ by permutations $\Sigma_2 \cong \mathbb{Z}_2$. Thus:

$$\mathrm{Char}(\pi, SO(4,\mathbb{C})) \cong \mathrm{Sym}^2\Big(\mathrm{Char}(\pi, SL(2,\mathbb{C}))\Big) \tag{6.3.22}$$

We have established before that $\mathrm{Char}(\pi, SL(2,\mathbb{C}))$ is a union of a trident of complex lines $\mathbf{Y}_{\mathbb{C}}$ and three isolated points[4], Thus:

$$\mathrm{Char}^0(\pi, SO(4,\mathbb{C})) \cong \mathrm{Sym}^2(\mathbf{Y}_{\mathbb{C}}) \tag{6.3.23}$$

[4]Recall the isolated points parametrize certain rigid G_2-structures.

which is topologically just $\mathbf{Y}_{\mathbb{C}} \times \mathbf{Y}_{\mathbb{C}}$.

Given the accordance with the previous computation, one must conclude that the map $\overline{\kappa}$ above is indeed surjective. We now discuss the geometric meaning of this condition. Recall that we introduced in Chapter 2 the notion of geometric structures in the sense of Goldman [Gol88]. Here, we are dealing with a $SO(4, \mathbb{C})$-structure over $\mathcal{G}_6$. We want to determine when representations $\rho : \pi_(\mathcal{G}_6) \to SO(4, \mathbb{C})$ can be lifted to representations $\widetilde{\rho} : \pi_1(\mathcal{G}_6) \to SL(2, \mathbb{C})^2$ into the universal cover $SL(2, \mathbb{C})^2$ of $SO(4, \mathbb{C})$. Indeed, such a lift is equivalent to surjectivity of $\overline{\kappa}$.

In [Cul86] a condition is determined for such a lift to exist. In fact, the result applies to any covering space, not just the universal cover. The condition can be described as follows: consider $SO(4) \subset SO(4, \mathbb{C})$ a maximal compact subgroup. We see the representation ρ as the holonomy of a $SO(4, \mathbb{C})/SO(4)$-structure on $\mathcal{G}_6$. One way this can be defined is via an immersion of the universal cover $\widetilde{G_6} \cong \mathbb{R}^3$ into $P := SO(4, \mathbb{C})/SO(4)$. We have a natural $SO(4)$-bundle over P, namely, $SO(4, \mathbb{C})$, which can then be pulled-back by the immersion to give a $SO(4)$-bundle E over $\mathbb{R}^3$. This bundle is of course trivial. Moreover, it has a properly discontinuous action of $\pi_1(\mathcal{G}_6)$ taking fibers to fibers, and the quotient $F := E/\pi_1(\mathcal{G}_6)$ defines a $SO(4)$-bundle over $\mathcal{G}_6$. Culler proves that if F has a section, then ρ lifts to any covering group of $SO(4, \mathbb{C})$. It is not clear to the author how the existence of flat $SO(4, \mathbb{C})$-Higgs fields on $\mathcal{G}_6$ is related to the existence of such section, but given the results discussed, presumably a relationship should exist.

6.4 A new proposal for SYZ

The SYZ conjecture proposes to construct the mirror as a dual special Lagrangian fibration where the mirror fibers are moduli spaces of $U(1)$-local systems of the original fibers (i.e., dual tori).

The computations in the last section suggest a new geometric construction of SYZ mirrors, which should prove useful at least whenever there are natural covering maps from the smooth fibers to a singular fiber. We write it as a conjecture:

Conjecture 6.4.1. *Let X be a non-compact threefold endowed with a special Lagrangian torus fibration $f : X \to B$ with generic smooth fiber $\boldsymbol{T}$ and a semi-flat Calabi-Yau metric. Fix a point $0 \in \mathcal{B}$ such that $F_0 := f^{-1}(0)$ is a singular fiber. Assume there is a covering map $\boldsymbol{T} \to F_0$ with finite Galois group $\Gamma \leqslant SU(2)$, and let G_Γ be the complex semisimple Lie group McKay-dual to Γ.*

Consider the Hitchin map $\mathscr{H} : \mathcal{M}^{G_\Gamma}_{Higgs} \to \mathcal{B}$. Then the mirror special Lagrangian fibration $\widehat{f} : \widehat{X} \to B$ has a crepant resolution given by:

$$\mathscr{H}|_{\check{X}} : \check{X} \to \mathscr{H}(\check{X}) \tag{6.4.1}$$

where $\check{X} := \mathcal{M}^{\Gamma}_{Higgs} \subset \mathcal{M}^{G_\Gamma}_{Higgs}$, the spectral mirror of X, is the locus of flat Higgs bundles whose spectral cover has Galois group Γ.

Moreover, by Theorem 5.1.7, $\check{X}$ is a moduli space of G_Γ-configurations of A-branes wrapping $F_0 \subset X$.

Chapter 7

Future Directions

7.1 Heterotic duals

It is well-known that M-theory on a $K3$ surface is dual to $E_8 \times E_8$-heterotic strings on $\mathbf{T}^3$. The duality is expected to hold when the two sides are appropriately fibered over a 3-manifold Q.

On the M-theory side, the theory is compactified on a G_2-manifold M, and the bundle $M \to Q$ is required to be a coassociative $K3$ fibration. The moduli space $\mathcal{M}_{G_2}^{\mathbb{C}}$ of the theory parametrizes complexified G_2-structures on M. In previous chapters we described the "hyperkähler sector"[1] of $\mathcal{M}_{G_2}^{\mathbb{C}}$ explicitly for ADE G_2-platyfolds.

On the heterotic side, we compactify the theory on a Calabi-Yau manifold X. The moduli space $\mathcal{M}_{\mathrm{Het}}$ is parametrized by a choice of special Lagrangian torus

[1] I.e., the G_2-structures coming from hyperkähler deformations of the fibers.

fibration $X \to Q$ and a choice of flat bundle $\mathcal{E}_q$ for each special Lagrangian fiber $T_q \cong \mathbf{T}^3$. The bundles E_q are called the "gauge bundles" of the theory. In general, the hyperkähler sector on the M-theory side will be mapped to a nontrivial subspace $\mathcal{S} \subseteq \mathcal{M}_{\text{Het}}$ - i.e., neither the special Lagrangian fibration nor the family of gauge bundles are constant on $\mathcal{S}$.

However, there is a suitable limit on M where the setup simplifies: this is when the $K3$ fibers degenerate into two "half-$K3$" surfaces $\frac{1}{2}K3_1$ and $\frac{1}{2}K3_2$ connected by a "long neck" isomorphic to $T^3 \times [0,1]$ [Mor02] [BSN17]. Recall that a generic $K3$ surface fibers elliptically over $\mathbb{P}^1$ with 24 singular fibers. If z is the coordinate on $\mathbb{P}^1$, the fibration can be described by $y^2 = x^3 + f(z)x + g(z)$ where f and g are polynomials with $\deg(f) \leqslant 8$, $\deg(g) \leqslant 12$. The singular fibers are located at the *discriminant locus* $\Delta = \{4f^3(z) + 27g^2(z) = 0\}$. The half-K3 surface is a rational elliptic surface that fibers over $\mathbb{P}^1$ with 12 singular fibers (generically). The limit is then a metric deformation of the Ricci-flat metric on $K3$ in which the long neck is created, separating the two sets of 12 singular fibers.

Now, remove from a half-$K3$ a smooth $\mathbf{T}^2$-fiber. Chen and Chen [CC16] prove that the resulting space is biholomorphic to the Tian-Yau *ALH-space* [TY90]. The ALH-spaces are non-compact hyperkähler manifolds very similar to the ALE-spaces we worked with in Chapter 3, the main difference being that the hyperkähler metric is asymptotic to $\mathbb{R} \times \mathbf{T}^3/\mathbb{Z}_2$, as opposed to $\mathbb{C}^2/\Gamma$. ALH spaces also satisfy a Torelli-type classification theorem: any ALH-space is diffeomorphic to the minimal resolution Z of $\mathbb{R}\times\mathbf{T}^3/\mathbb{Z}_2$, and any ALH hyperkähler structure comes from a choice of hyperkähler structure on Z. The main result regarding ALH-spaces in [CC16] is that one can glue two such spaces with a $\mathbf{T}^3\times[0,1]$ in between to produce a $K3$ surface. For further reference, we note that the moduli space of hyperkähler metrics on Z is 33-dimensional.

From the point of view of the heterotic side of the duality, the $\frac{1}{2}K3$ limit is interesting because it corresponds to a regime in which the volume of the special Lagrangian torus fibers $\mathbf{T}^3$ of $X\to Q$ is large and the duality data "decouples": the monodromies of the $K3$-lattice in the neck are matched with the SYZ fibration data $X\to Q$, and the Ricci-flat metric on the two $\frac{1}{2}K3$'s is matched with the flat $E_8\times E_8$-bundle on $\mathbf{T}^3$ [BSN17]. One can think of each $\frac{1}{2}K3$ as producing a flat E_8-bundle.

It is not hard to prove that the moduli of Ricci-flat metrics on a $\frac{1}{2}K3$ is the same as the moduli of flat E_8-bundles on $\mathbf{T}^3$ [Mor02]; they are both 24-dimensional real tori, and the proof that it is the same torus essentially boils down to a Mayer-Vietoris argument to determine the $\frac{1}{2}K3$-lattice and its intersection form. Now, if we introduce an ADE singularity $\mathbb{C}^2/\Gamma$ of type G in the half-K3, then locally around it, deformations of the Ricci-flat metric are given by deformations of the hyperkähler structure of the ALE $\mathbb{C}^2/\Gamma$. This introduces $\dim(\mathfrak{h}_c)$ extra parameters. On the heterotic side, this induces a reduction of the structure group of the gauge bundle to $C_{E_8}(G)$. It can also be seen as a choice of flat G-connection whose holonomy commutes with the gauge connection. When we fiber this over a 3-manifold Q, we expect the hyperkähler sector of the G_2-deformations to match "families of flat G-connections on T^3" over Q.

We now proceed to prove this last assertion rigorously when Q is a platycosm. Throughout this section we work over the smooth locus of the fibration $X \to Q$.

For each T_q, a choice of flat G-bundle corresponds to a point of the character variety

$$\mathrm{Char}(T_q, G) := \mathrm{Hom}(\pi_1(T_q), G)/\mathcal{C}_G \tag{7.1.1}$$

We would like to argue that a choice of heterotic modulus is a certain section of a flat bundle:

$$\begin{tikzcd} \mathrm{Char}(T_q, G) \arrow[r, hook] & \mathcal{E} \arrow[d] \\ & Q \end{tikzcd}$$

We first construct the flat bundle structure, and then argue what kind of section we want.

First, recall that X is taken to be a special Lagrangian torus fibration over the flat manifold Q. It is natural to require the fibration to be compatible with the fixed flat structure on Q. We interpret this as requiring that $X \to Q$ is induced from a δ-flat family of lattices $\Lambda_q \subset T_q^*Q$ by $X \cong T^*Q/\underline{\Lambda}$, where $\underline{\Lambda} = \bigcup_{q \in Q} \Lambda_q$ In particular, the structure group of $X \to Q$ is taken to be the isometry group of $\underline{\Lambda}$, i.e., $\mathrm{Aff}(\underline{\Lambda})$. In particular, $X \to Q$ is *not* a vector bundle.

The group $\mathrm{Aff}(\underline{\Lambda})$ fits into an exact sequence:

$$1 \to (\mathbf{S}^1)^3 \to \mathrm{Aff}(\underline{\Lambda}) \to \mathbb{M}(\mathbb{R}^3/\underline{\Lambda}) \to 1 \tag{7.1.2}$$

where $\mathbb{M}(\mathbb{R}^3/\underline{\Lambda}) \cong SL(3,\mathbb{Z})$ is the mapping class group of the three-torus $\mathbf{T} \cong \mathbb{R}^3/\underline{\Lambda}$.

Now, the action of $\mathrm{Aff}(\underline{\Lambda})$ on $\mathbf{T}$ descends to an action of $\mathbb{M}(\mathbf{T})$ on $\pi_1(\mathbf{T})$. This is because translations act trivially on homotopy classes, and $\mathbb{M}(\mathbf{T}) \cong Out^+(\mathbf{T})$. This action dualizes to an action on $\mathrm{Hom}(\mathbf{T}, G)$, and the new action is well-defined on conjugacy classes; hence it descends to $\mathrm{Char}(T_q, G)$. This gives $\mathcal{E}$ the structure of a $SL(3,\mathbb{Z})$-bundle over Q.

To define a flat connection on $\mathcal{E} \to Q$, we first define one on $X \to Q$. We have the flat connection δ on $T^*Q \to Q$. Since $\underline{\Lambda}$ is taken to be δ-flat, δ induces a flat connection on $X \to Q$. Using the short exact sequence above, one can induce a flat connection on a $\pi_1(\mathbf{T})$-bundle, which can then be dualized to a flat connection on $\mathcal{E} \to Q$, as desired.

We are interested in describing the flat sections of $\mathcal{E}$, i.e., the moduli space is:

$$H^0_{\text{flat}}(Q, \mathcal{E}) \tag{7.1.3}$$

A flat section can be determined by solving the parallel transport equation: we fix a fiber $\mathcal{E}_q$ and $s_0 \in \mathcal{E}_q$ and require $\nabla s = 0$, $s(q) = s_0$. Then s is determined at any other fiber $\mathcal{E}_p$ by lifting a path from q to p to its unique horizontal lift. This gives a multivalued map from horizontal lifts to a fiber. The map is multivalued because the horizontal lift is only unique modulo monodromy: different points in a fiber can give rise to the same flat section, and such points are related by the action of the monodromy group M_∇, which should be thought as a global symmetry.

However, when restricted to flat sections of $\mathcal{E}$, the map becomes single-valued and also a bijection. The flat sections are exactly the lifts that are fixed by the monodromy. Hence the moduli space can be described as:

$$\mathcal{M}_{\text{Het}} = H^0_{\text{flat}}(Q, \mathcal{E})^{M_\nabla} \tag{7.1.4}$$

Equivalently, $\mathcal{M}_{\text{Het}}$ can be obtained by taking the fiberwise quotient of $\mathcal{E}$ by M_∇. The resulting bundle is the *holonomy bundle* $\mathcal{H}_\nabla$ *associated to* $(\mathcal{E}, \nabla)$. It has a natural flat connection induced from ∇. The moduli space is:

$$\mathcal{M}_{\text{Het}} \cong H^0_{\text{flat}}(Q, \mathcal{H}_\nabla) \tag{7.1.5}$$

Under the correspondence between flat sections and fiber, we get a more useful description:

$$\mathcal{M}_{\text{Het}} \cong \text{Char}(\mathbf{T}^3, G)^{M_\nabla} \tag{7.1.6}$$

But notice this is exactly the answer we got for the moduli space of classical A-branes in the type IIA picture! We have proved the following:

Theorem 7.1.1. *Let $M \to Q$ be a platycosm ADE G_2-orbifold of type G. Let $\mathcal{M}_A$ be the moduli space of classical A-branes on its type IIA dual T^*Q (i.e., flat G-*

connections on Q), and let $\mathcal{M}_{Het}$ be the moduli space of its heterotic dual $X \to Q$, parametrizing flat families of flat bundles on the sLag tori X_q. Then:

$$\mathcal{M}_A \cong \mathcal{M}_{Het} \cong \mathit{Char}(\boldsymbol{T}^3, G)^{M_\nabla} \tag{7.1.7}$$

In particular, for $G = SL(n, \mathbb{C})$ these moduli spaces are:

$$\left((\mathbb{C}^*)^{3n-3} \Big/ S_n \right)^{M_\nabla} \tag{7.1.8}$$

These have been computed for $Q = T^3$ and the Hantzsche-Wendt manifold $\mathcal{G}_6$ in a previous section.

7.2 G_2 Intermediate Jacobian and Variation of Hodge Structures

Let M be a compact oriented G_2-manifold, and let $\star$ denote the usual Hodge star operator induced from the G_2-metric. We will use the notation $H^k_A := H^k(M, A)$. The moduli space of G_2-structures on M is an open set inside $H^3(M, \mathbb{R})$ and is denoted by $\mathcal{M}_{G_2}$. We call $\mathcal{M}^{\mathbb{C}}_{G_2}$ its "complexification", which is a Lagrangian torus fibration over $\mathcal{M}_{G_2}$ admitting a natural Lorentzian Kähler metric [KL07]. Let f be the Kähler potential for this metric.

Consider the two real tori:

$$J_1 = {(H^3 \oplus H^4)_{\mathbb{R}}}\Big/{(1+\star)H^3_{\mathbb{R}} \oplus (1-\star)H^3_{\mathbb{Z}}} \cong {H^4_{\mathbb{R}}}\Big/{\star H^3_{\mathbb{Z}}} \tag{7.2.1}$$

$$J_2 = {(H^4 \oplus H^3)_{\mathbb{R}}}\Big/{(1+\star)H^4_{\mathbb{R}} \oplus (1-\star)H^4_{\mathbb{Z}}} \cong {H^3_{\mathbb{R}}}\Big/{\star H^4_{\mathbb{Z}}} \tag{7.2.2}$$

Unlike the intermediate Jacobians of a compact Kähler manifold[2], these tori can be odd dimensional.

In this section we will study "complexifications" of these tori. The main point is that in such a situation one can use techniques of Hodge theory. The goal is to determine if the complex tori can be assembled into a family with the structure of a complex integrable system.

7.2.1 Complex Tori

There are three complex tori that can be constructed from J_1 and J_2. The first two are:

$$J' = J_1 \times J_1 \cong {H^4_{\mathbb{C}}}\Big/{\star H^3_{\mathbb{Z}} \oplus \star H^3_{\mathbb{Z}}} \tag{7.2.3}$$

$$J'' = J_2 \times J_2 \cong {H^3_{\mathbb{C}}}\Big/{\star H^4_{\mathbb{Z}} \oplus \star H^4_{\mathbb{Z}}} \tag{7.2.4}$$

endowed with the obvious complex structures.

[2]The Hodge decomposition implies that odd Betti numbers of compact Kähler manifolds are even.

Let J be the torus defined by:

$$J = J_1 \times J_2 \cong (H^3 \oplus H^4)_{\mathbb{R}} \Big/ \star H^4_{\mathbb{Z}} \oplus \star H^3_{\mathbb{Z}} \tag{7.2.5}$$

Proposition 7.2.1. *J' and J'' are dual complex tori, while J is self-dual.*

Proof. This is a simple application of Poincaré duality. $\square$

Given dual complex tori T and $\hat{T}$, a non-degenerate line bundle L on X defines an isogeny ψ_L of degree $\det(c_1(L))$ by the formula $\psi_L(x) = t_x^* L \otimes L^{-1}$, where $t_x : T \to T$ is translation by x [Deb99]. As a consequence, every non-degenerate line bundle L' on J' defines an isogeny $\psi_{L'} : J' \to J''$ of degree $\det(c_1(L'))$. Similarly, a line bundle L on J defines a self-isogeny $\psi_L : J \to J$ of degree $\det(c_1(L))$. There are Poincaré line bundles $\mathcal{P}' \to J' \times J''$ and $\mathcal{P} \to J \times J$.

We will focus on J as opposed to J' or J'', but we note that all results that follow have analogous versions for the other tori. We also remark that the lattices defining all three tori depend on the G_2-structure φ through $\star$.

We have not yet fixed an isomorphism $(H^3 \oplus H^4)_{\mathbb{R}} \cong \mathbb{C}^{b_3}$, and the statements about J only make sense once one is fixed. Consider the map:

$$I : (H^3 \oplus H^4)_{\mathbb{R}} \to (H^3 \oplus H^4)_{\mathbb{R}}$$

$$(\eta, \theta) \mapsto (\star\theta, - \star \eta)$$

Since $\star^2 = 1$, this is a complex structure. Notice that it differs from those of

J' and J'' in that it "sees" the G_2-structure on M even at the level of $\mathbb{C}^{b_3}$, via the Hodge star $\star$.

From now on we use the notation $H_A := (H^3 \oplus H^4)_A$. When we go to the complexification $H_{\mathbb{C}} := H_{\mathbb{R}} \otimes \mathbb{C}$, the i-eigenspace of I is $(1 + i\star)H_{\mathbb{R}}$. Therefore I coincides with the obvious complex structure on H given by the operator $i\star$ - whose i-eigenspace is $\mathcal{V} := (1 + \star)H_{\mathbb{C}}$. Therefore:

$$J \cong {}^{H_{\mathbb{C}}}\!/_{\mathcal{V}} \oplus \star H_{\mathbb{Z}} \tag{7.2.6}$$

$H_{\mathbb{C}}$ has a complex conjugation given by $\sigma : H_{\mathbb{C}} \to H_{\mathbb{C}}$, $\sigma(\eta, \theta) = (i \star \theta, -i \star \eta)$. It is easy to check that $\sigma^2 = id$ and $\sigma \circ I = -I \circ \sigma$. Complex conjugation of vector spaces will be taken with respect to σ. It is easy to see that $\overline{\mathcal{V}} = (1 - \star)H_{\mathbb{C}}$. Note also that $\mathcal{V} \oplus \overline{\mathcal{V}} = H_{\mathbb{C}}$.

Lemma 7.2.2. *The map I induces a complex structure on J.*

Proof. All we need to show is that the map descends to a map on the tangent bundle $T_{\mathbb{R}}J$, which is real-isomorphic to the quotient $H_{\mathbb{C}}/\mathcal{V} \cong \overline{\mathcal{V}}$. Suppose $(\eta, \theta) = (\mu, \nu) + (\alpha, \star\alpha)$. Then $I(\eta - \mu, \theta - \nu) = I(\alpha, \star\alpha) = (i\star\alpha, \star(i\star\alpha)) \in \mathcal{V}$, so $I\overline{(\eta, \theta)} = I\overline{(\mu, \nu)}$. $\square$

The complex structure I depends on the choice of G_2-structure on M through the Hodge star $\star$.[3] Moreover, I is completely determined by any of its *period matrices*. Also notice that $H_1(J, \mathbb{C}) \cong T_{\mathbb{C}}J \cong T^{1,0} \oplus T^{0,1}$, using the complex structure I. These are the eigenspaces associated to i and $-i$, so $H^1(J, \mathbb{C}) \cong \mathcal{V} \oplus \overline{\mathcal{V}}$.

[3]The G_2-structure φ determines a G_2-metric g_φ, which in turn determines $\star$.

Remark: The definition of the complex torus J is inspired by the *Lazzeri Intermediate Jacobian* of a compact oriented Riemannian manifold of dimension $2(2k+1)$ [BL99]. However, the Lazzeri Jacobian is an abelian variety, and as we will se below this is not true for J.

We define a weight 1 Hodge structure on $H_J := H^1(J, \mathbb{Z})$ by:

$$H_J = F^1 H_J \oplus \overline{F^1 H_J} \tag{7.2.7}$$

where $F^1 H_J := \mathcal{V}$ and $\overline{F^1 H_J} = \overline{\mathcal{V}}$. This makes sense since $T_{\mathbb{R}} J \cong \overline{\mathcal{V}}$ and $T_{\mathbb{C}} J \cong H_{\mathbb{C}}$ (non-canonically). Define $F^0 H_J = \overline{F^0 H_J} = H_J$, so $F^1 H_J = H_J^{1,0}$ and $\overline{F^1 H_J} = H_J^{0,1}$.

Consider the bilinear form:

$$Q : H_J \times H_J \to \mathbb{Z}$$
$$Q((\eta_1, \theta_1), (\eta_2, \theta_2)) = \int_M \eta_1 \wedge \theta_2 - \int_M \eta_2 \wedge \theta_1 \tag{7.2.8}$$

Lemma 7.2.3. *Q is a skew-symmetric form with the following properties:*

1. $Q(H_J^{1,0}, H_J^{1,0}) = 0 = Q(H_J^{0,1}, H_J^{0,1})$

2. $h((\eta_1, \theta_1), (\eta_2, \theta_2)) := iQ((\eta_1, \theta_1), \overline{(\eta_2, \theta_2)})$ *is a semi-Riemannian Kähler metric.*

Proof. 1) For $H_J^{1,0}$:

$$Q((\eta, \star\eta), (\nu, \star\nu)) = \int_M \eta \wedge \star\nu - \int_M \nu \wedge \star\eta = 0$$

by definition of $\star$. Similarly for $H_J^{0,1}$.

2) We use the definition of σ to show:

$$h((\eta_1, \theta_1), (\eta_2, \theta_2)) = \int_M \eta_1 \wedge \star\eta_2 + \int_M \star\theta_2 \wedge \theta_1$$

which is an extension of the L^2-metric on $\mathcal{M}_{G_2}$ to $\mathcal{M}_{G_2}^{\mathbb{C}}$. It is clearly symmetric and non-degenerate. In fact, it is the Lorentzian Kähler metric on $\mathcal{M}_{G_2}^{\mathbb{C}}$ described by Karigiannis and Leung in [KL07]. □

Corollary 7.2.4. *$(H_{\mathbb{Z}}, H_{\mathbb{C}}, Q)$ defines a polarized Hodge structure of weight 1.*

The polarized Hodge structure descends to J:

Lemma 7.2.5. *h is a polarization on J.*

Proof. See [Bec18] Lemma 3. □

Proposition 7.2.6. *The polarized complex torus (J, h) is not an abelian variety.*

Proof. This is a consequence of Lemma 7.2.5 and part 2 of Lemma 7.2.3, which shows that h has index 1. □

We now generalize this picture to a family of complex tori $\pi : \mathcal{J} \to B$, where $B \subseteq \mathcal{M}_{G_2}^{\mathbb{C}}$ is an open set and each fiber $J_{\varphi_{\mathbb{C}}}$ is the polarized Jacobian for the complexified G_2-structure $\varphi_{\mathbb{C}} \in B$. There is a locally constant sheaf $\mathcal{H}^{\mathbb{Z}} := R^1\pi_*\underline{\mathbb{Z}}$ over B whose stalk at each $\varphi_{\mathbb{C}}$ is $H^1(J_{\varphi_{\mathbb{C}}}, \mathbb{Z})$. The polarizations determine a map $Q : \mathcal{H}^{\mathbb{Z}} \otimes \mathcal{H}^{\mathbb{Z}} \to \underline{\mathbb{Z}}$. Associated to this local system there is a holomorphic bundle

$\mathcal{H} := \mathcal{H}^{\mathbb{Z}} \otimes \mathcal{O}_B$ and a flat holomorphic connection ∇, the Gauss-Manin connection. Griffiths transversality shows that the subbundle $\mathcal{F}^1$ with fibers $F^1H^1(J_{\varphi_{\mathbb{C}}}, \mathbb{C})$ is holomorphic. We have proved the following:

Theorem 7.2.7. *The data $(\mathcal{H}^{\mathbb{Z}}, \mathcal{F}^1, Q)$ defines a polarized variation of Hodge structures of weight 1.*

Given this result, we would like to know under which condition on the VHS, a C^∞ locally trivial family of tori $\pi : \mathcal{J} \to B$ admits a structure of *complex* integrable system, i.e., an analytically locally trivial structure on π with a compatible Poisson structure under which π is a Lagrangian fibration[4]. If we do not impose the analyticity condition, then there are no obstructions: any C^∞ fibration can be given local action-angle coordinates, which in turn define a Poisson structure.

One way to approach this problem is to produce a *Seiberg-Witten differential*. However, this is a rather strong condition, as it implies that the total space is an exact symplectic space. One can always find a Seiberg-Witten 1-form in the relative universal cover to $\mathcal{J} \to B$, and its differential will be a symplectic form that descends to the base, even though the Seiberg-Witten form only does so locally. This is an interesting computation. but we will approach the problem differently.

What we need is the *cubic condition* of Donagi and Markman [DM95]: in its local form, it states that given the classifying map $q : \mathcal{J} \to \mathcal{C}_{b_3,1}$, where $\mathcal{C}_{b_3,1}$ is the moduli space of polarized complex tori of dimension b_3 and index 1, and given

[4]Note that this is weaker than an *algebraic* integrable system. See [Bec18], Definition 3.

an isomorphism $\tau : \mathcal{V}^* \to TB$, where $\mathcal{V}$ is the vertical bundle of $\mathcal{J}$, we need the composition

$$dq \circ \tau : \mathcal{V}^* \to Sym^2(\mathcal{V}) \tag{7.2.9}$$

- which is an element of $\Gamma(\mathcal{V} \otimes \mathrm{Sym}^2(\mathcal{V}))$ - to be given by a cubic $c \in \Gamma(\mathrm{Sym}^3(\mathcal{V}))$.

Moreover, Donagi and Markman prove a *global* cubic condition: if there is a *holomorphic* function $F : B \to \mathbb{C}$ such that $q = \frac{\partial^2 F}{\partial z_i \partial z_j}$ then $\mathcal{J} \to B$ is a Lagrangian fibration with cubic given by $\frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k}$.

Therefore, our problem reduces to finding a natural holomorphic function on $\mathcal{M}^{\mathbb{C}}_{G_2}$: by the global cubic condition this will automatically fix the Lagrangian structure and the classifying map of the family of tori.

Karigiannis-Leung [KL07], Grigorian-Yau [GY08] proved there is a natural cubic form on $\mathcal{M}_{G_2}$, called the Yukawa coupling. It is given by a real-analytic function:

$$f(\varphi) = \frac{3}{7} \int_M \varphi \wedge \star\varphi = 3 \int_M dvol_\varphi \tag{7.2.10}$$

It can be extended to a complex function on B that is constant along the fibers of $\mathcal{J}$. However, this extension is not holomorphic, hence not fit for our purposes. Instead, we work near a smooth point in $\mathcal{M}_{G_2}$ and fix an open set $U \subseteq \mathcal{M}_{G_2}$ where f can be extended in a power series. Recall that $\mathcal{M}_{G_2}$ is an open set in $H^3(M, \mathbb{R})$, so $\dim_{\mathbb{R}} U = b_3$.

Let $f|_U : U \to \mathbb{R}$, and let $W \subseteq \mathbb{C}^{b_3}$ be the domain of holomorphy of f. Let

$\overline{f} : W \to \mathbb{C}$ be the extension of f to a holomorphic function. We need to show the following:

Conjecture 7.2.8. *$\overline{f}$ descends to a holomorphic function $F : B \to \mathbb{C}$. In other words, $\overline{f}$ is $\star H^3_{\mathbb{Z}}$-periodic on the imaginary directions.*

The idea for the proof is to show that F is basically obtained by modifying formula 7.2.10 by redefining φ to include the holonomies of the C-field. It would be interesting to prove the conjecture above as it would tie up the Lagrangian structures on $\mathcal{J} \to \mathcal{M}^{\mathbb{C}}_{G_2}$ and $\mathcal{M}^{\mathbb{C}}_{G_2} \to \mathcal{M}_{G_2}$.

Another path to produce a periodic holomorphic function on $\mathcal{M}^{\mathbb{C}}_{G_2}$ is the following. Fix a class $\gamma \in (\star H^4(M, \mathbb{Z}))^*$. Over $\mathcal{M}^{\mathbb{C}}_{G_2}$, this defines a locally flat section of integration cycles. The idea is to define local functions $F_i : U_i \subset B \to \mathbb{C}$ over a trivializing cover for the flat sheaf by:

$$F_i(\varphi + iC) = \int_{\gamma(U_i)} \exp(\varphi + iC) \tag{7.2.11}$$

and then glue them together to a global holomorphic function $F : B \to \mathbb{C}$.

Recall that $\varphi_{\mathbb{C}} := \varphi + iC$ is the natural holomorphic coordinate on $\mathcal{M}^{\mathbb{C}}_{G_2}$, so F is holomorphic. Hence if F can be constructed, it will have all the desired properties: it is holomorphic and periodic in the imaginary directions, hence well-defined on $\mathcal{M}^{\mathbb{C}}_{G_2}$. So it can be taken as a holomorphic potential $F : \mathcal{M}^{\mathbb{C}}_{G_2} \to \mathbb{C}$ giving $\mathcal{J} \to \mathcal{M}^{\mathbb{C}}_{G_2}$ the structure of a complex integrable system.

7.2.2 Concerning special Kähler structures

An interesting question is whether one can have non-trivial G_2-manifolds such that the Kähler metric on $\mathcal{M}^{\mathbb{C}}_{G_2}$ admits an adapted special Kähler structure, that is, a flat torsion-free symplectic connection ∇ satisfying $d_\nabla I = 0$. The reason this is plausible is that $\mathcal{M}^{\mathbb{C}}_{G_2}$ already comes equipped with the following structures:

Lemma 7.2.9. *There is a flat torsion-free connection ∇ on $T\mathcal{M}^{\mathbb{C}}_{G_2}$ satisfying the following conditions:*

- *the Kähler metric is the Hessian of a Kähler potential with respect to a ∇-flat holomorphic coordinate system.*
- *there is a holomorphic cubic form $\mathcal{Y} \in H^0(\mathcal{M}^{\mathbb{C}}_{G_2}, Sym^3(T^*\mathcal{M}^{\mathbb{C}}_{G_2}))$ which determines the Christoffel symbols of ∇ (see the definition of the tensor B below).*

Proof. See [GY08], equations 6.36 and 6.37. Basically ∇ is an extension to $T\mathcal{M}^{\mathbb{C}}_{G_2}$ of the covariant derivative on the canonical real line bundle $L_\varphi \to \mathcal{M}^{\mathbb{C}}_{G_2}$ defined by equation 6.32 in [GY08]. □

However, this is not enough to produce a special-Kähler structure. We need further compatibility conditions between $\mathcal{Y}$, ∇ and the Kähler structure. More precisely: let $B \in \Omega^{1,0}(\mathcal{M}^{\mathbb{C}}_{G_2}, \mathrm{End}_{\mathbb{R}} T\mathcal{M}^{\mathbb{C}}_{G_2})$ be defined by:

$$\mathcal{Y} = -\omega(\pi^{1,0}, [B, \pi^{1,0}]) \tag{7.2.12}$$

The first condition is that

$$\nabla = D + B \tag{7.2.13}$$

where D is the Levi-Civita connection for the Kähler metric. This is shown in [GY08], equation 6.36 (in that paper, B is denoted by A^N_{MQ}, and $\mathcal{Y}$ by A_{MNQ}).

However, there are further conditions:

$$d_D B = 0$$

$$F_D + \left\{B \wedge \overline{B}\right\} = 0 \tag{7.2.14}$$

Here, $\{\cdot, \cdot\}$ is the wedge-anti-commutator. Formula 6.22 in [GY08] shows the second condition is not true in general, but it is possible that only the second term survives for some specific topological types of G_2-manifolds. Essentially, the issue is that $\mathcal{Y}$ depends on the G_2-structure φ, while F_D depends on the fourth derivative of the Kähler potential, which depends on $\star\varphi$. This issue does not arise in the moduli space of Calabi-Yau manifolds, as both structures come from the holomorphic 3-form.[5]

The cotangent bundle of a special Kähler manifold is automatically hyperkähler, and a flat cotangent lattice Λ defines an algebraic integrable system $(T^*\mathcal{M}^{\mathbb{C}}_{G_2})/\Lambda \to \mathcal{M}^{\mathbb{C}}_{G_2}$ with a family of polarizations making the fibers into abelian varieties. For this reason, such integrable systems are more special than those discussed in the previous section, so it would be interesting to determine which topological types of G_2-manifolds admit this extra structure on the moduli space $\mathcal{M}^{\mathbb{C}}_{G_2}$.

[5] I thank Sergey Grigorian for explaining this point to me.

7.3 Kapustin-Witten systems and flat Higgs bundles

In this section we change gears in order to discuss flat Higgs bundles in the context of a larger theory: Kapustin-Witten systems on a four-manifold.

We show that flat Higgs bundles describe an invariant subspace of a natural map on the moduli space of solutions of the Kapustin-Witten equations. More generally, we argue that the moduli space of a family of extended Bogomolny theories parametrized by a parameter $q \in \mathbf{S}^1$ can be given the structure of a $\mathbf{S}^1$-bundle over the moduli space of solutions to the Acharya-Pantev-Wijnholt system.

Let $M = Q \times \mathbb{R}$ be an oriented Riemannian four-manifold, $\mathcal{A}$ a connection on a G-bundle on $E \to M$, $\mathcal{F}$ its curvature, and $\Phi \in \Omega^1(Ad(E))$. Consider the Kapustin-Witten equations [KW06]:

$$\begin{aligned}
(\mathcal{F} - \Phi \wedge \Phi + q D_{\mathcal{A}} \Phi)^+ &= 0 \\
(\mathcal{F} - \Phi \wedge \Phi - q^{-1} D_{\mathcal{A}} \Phi)^- &= 0 \\
D_{\mathcal{A}} \star_M \Phi &= 0
\end{aligned} \tag{7.3.1}$$

where for a two-form α, $\alpha^{\pm}$ denotes its selfdual and anti-selfdual components, $\star_M$ is the Hodge star operator on M and $q \in \mathbb{R}$.

We refer to 7.3.1 as the KW_q *system* and q is called a twisting parameter. In [GW11], Gaiotto and Witten study the dimensional reduction of the KW_1 system down to two dimensions, and show the moduli space of solutions provides knot invariants. In section 7 below I will sketch an approach to relate the discussion on this section to [GW11] and also the work of Aganagic and Vafa [AV01].

The KW_q system is a deformation of the Hermitian-Yang-Mills equations. We proved in section 4 that the APW system is a dimensional reduction of HYM. Hence APW is also a dimensional reduction of KW_q. The precise way in which this happens is given by the following lemma:

Lemma 7.3.1. *Let $p_Q : M \to Q$ be the projection to Q. The, for $q \neq \pm i$, the pullback of connections by p_Q defines an injective map $\iota : \mathcal{M}_{APW}(Q) \hookrightarrow \mathcal{M}_{KW_q}(M)$*

Proof. Start from 7.3.1 and perform dimensional reduction, assuming moreover that $\mathcal{A} = \pi^* A$ and $\Phi = \pi^* \phi$ (i.e., $A_0 = \phi_0 = 0$). Let's see what happens with the first equation: let $\mathcal{G} = \mathcal{F} - \Phi \wedge \Phi + q D_{\mathcal{A}} \Phi$. Then the equation is:

$$\mathcal{G} = - \star_M \mathcal{G} \tag{7.3.2}$$

But $\mathcal{G}$ has no dt components, and $\star_M \mathcal{G}$ has only components with a dt, so they must both vanish. Thus we get:

$$\mathcal{F} - \Phi \wedge \Phi + q D_{\mathcal{A}} \Phi = 0 \tag{7.3.3}$$

Now note that everything is pulled-back from Q. So:

$$F - \phi \wedge \phi + q D_A \phi = 0 \tag{7.3.4}$$

An analogous argument holds for the other two equations. We get:

$$F - \phi \wedge \phi + qD_A\phi = 0$$

$$F - \phi \wedge \phi - q^{-1}D_A\phi = 0$$

$$D_A \star \phi = 0 \tag{7.3.5}$$

If $q \neq \pm i$, the first and second equations combined imply that $D_A\phi = 0$. Therefore we obtain the flat Higgs bundles equations. $\square$

So far this is hardly interesting, as it is well-known that most of the solutions to KW_q are flat connections. The interesting idea comes when we compare the dimensionally reduced KW_q for different values of q. For $q = 1$, we get equations 10.35 in [KW06]:

$$F - [\phi, \phi] = \star\Big(D_A\phi_0 - [A_0, \phi]\Big)$$
$$D_A\phi = \star\Big(D_AA_0 + [\phi_0, \phi]\Big)$$
$$D_A \star \phi = \star[A_0, \phi_0] \tag{7.3.6}$$

We now perform the dimensional reduction of the KW_0 system. The equations are:

$$(\mathcal{F} - \Phi \wedge \Phi)^+ = 0$$

$$(D_{\mathcal{A}}\Phi)^- = 0$$

$$D_{\mathcal{A}} \star \Phi = 0 \tag{7.3.7}$$

Let t be a coordinate on $\mathbb{R}$, $\{x_1, x_2, x_3\}$ coordinates on Q and write $\mathcal{A} = A_0 dt + A_i dx^i$, $\Phi = \phi_0 dt + \phi_i dx^i$.[6] We write $A = A_i dx^i$ and $\phi = \phi_i dx^i$ and think of these as a connection and a Higgs field on Q, respectively. Let F be the curvature of A.

We assume the matrices do not depend on t. The first equation in 7.3.7 becomes a set of three equations:

$$F_{ij} - [\phi_i, \phi_j] = (-1)^k \Big(D_k A_0 + [\phi_0, \phi_k] \Big) \qquad (i < j, i \neq k \neq j) \tag{7.3.8}$$

These can be rewritten using the Hodge star $\star$ on Q:

$$F - [\phi, \phi] = \star \Big(D_A A_0 + [\phi_0, \phi] \Big) \tag{7.3.9}$$

(Notice that A_0 and ϕ_0 are just matrix-valued functions on Q, so $D_A A_0$ and $[\phi_0, \phi]$ are matrix-valued one-forms on Q, and are mapped to matrix-valued two-forms by $\star$).

Next, the second equation in 7.3.7 becomes:

$$D_{A_i} \phi_j = [A_0, \phi_k] - D_{A_k} \phi_0 \qquad (i < j, i \neq k \neq j) \tag{7.3.10}$$

which can be put together as:

$$\star D_A \phi = [A_0, \phi] - D_A \phi_0 \tag{7.3.11}$$

[6] In this section we use the Einstein summation convention for upper and lower indices.

Finally, the last equation in 7.3.7 is equivalent to:

$$[A_0, \phi_0] dt dx^1 dx^2 dx^3 + (-1)^i D_{A_i} \phi_i = 0 \tag{7.3.12}$$

which can be rewritten as:

$$D_A \star \phi = \star [A_0, \phi_0] \tag{7.3.13}$$

Putting all together we get the following set of equations:

$$\begin{aligned} F - [\phi, \phi] &= \star\Big(D_A A_0 + [\phi_0, \phi] \Big) \\ D_A \phi &= \star\Big(- D_A \phi_0 + [A_0, \phi] \Big) \\ D_A \star \phi &= \star [A_0, \phi_0] \end{aligned} \tag{7.3.14}$$

for a G-connection A on $E \to Q$ with curvature F, an element $\phi \in \Omega^1(Q, Ad(E))$ and two fields $A_0, \phi_0 \in \Omega^0(Q, Ad(E))$. Notice we used the fact that $\star^2 = 1$.

When $\phi_0 = A_0 = 0$ both sets of equations 7.3.6 and 7.3.14 reduce to:

$$\begin{aligned} F - \phi \wedge \phi &= 0 \\ D_A \phi &= 0 \\ D_A \star \phi &= 0 \end{aligned} \tag{7.3.15}$$

which are exactly the Acharya-Pantev-Wijnholt (APW) equations.

Let KW_1 denote the space of solutions to 7.3.6 and KW_0 the space of solutions to 7.3.14. There is an "anti-involution":

$$KW_0 \to KW_1$$

$$(A_0, \phi_0) \to (\phi_0', -A_0') \tag{7.3.16}$$

that matches the APW subspaces of KW_1 and KW_0. These subspaces parametrize the same objects, so we do not distinguish them.

In fact, one can perform the dimensional reduction for general parameter q, and from the general formula one can see that q defines a $\mathbf{S}^1$-action on $KW := \sqcup_{q \in \mathbf{S}^1} KW_q$ leaving APW invariant. More precisely, if one considers the family over APW whose fiber over (A, ϕ, h) is $(q, A_0(q), \phi_0(q))$, then $\mathbf{S}^1$ acts by rotation on the fibers.

The relationship between KW and APW seems more surprising when one considers the analogy with instantons on four-manifolds. If 7.3.1 is analogous to the anti-selfduality equations on M, then 7.3.14 correspond to the Bogomolny equations on Q and 7.3.15 are analogous to the rather boring solutions given by flat connections on Q. Of course, in the classical situation one works with a real gauge group, while in the present case APW systems are actually *equivalent* to flat complex connections. Hence the analogy is stronger than it seems at first.

This raises a few interesting questions:

1. Does the family $KW \to APW$ admits more interesting geometric structures,

such as a natural connection?

2. In [PW11] Pantev and Wijnholt introduced a Morse-Novikov complex counting solutions to 7.3.15. In [Wit12], Witten argues that counting certain solutions to 7.3.1 should give rise to Khovanov homology. Can the Morse-Novikov homology be realized as a subcomplex, and if so, what kind of knot invariants does it give rise to?

   In general, we expect that APW is some sort of "invariant subspace" for a flow of time-independent solutions $\mathrm{KW}_q \to \mathrm{KW}_r$. In other words, if (2) holds, the Morse-Novikov complex should be an invariant subspace for the differentials computing Khovanov homology.

7.4 Other directions

1. G_2**-metrics:** The most immediate extension of this work is to identify subfamilies of the deformation family of closed G_2-structures constructed in Chapter 3 that correspond to G_2-metrics. In other words, we would like to know which fibers of the deformation family have the property that its induced closed G_2-structure is also torsion-free. For a given $s \in \mathcal{B} = \Gamma_{\text{flat}}(Q, \mathcal{E})$, this is equivalent to the harmonicity of a certain *Donaldson section* $\mathcal{D}_s : Q \to H^2(M_s/s(Q), \mathbb{R})$ of the flat bundle $H^2(M_s/s(Q), \mathbb{R})$ over Q [Don16]. This section is determined by the condition that the cohomology class of the hyperkähler ele-

ment η_s is given by the derivative of $\mathcal{D}_s$. An equivalent way to formulate the condition is that the image $\mathcal{D}_s(Q)$ must be a maximal submanifold of $H^2(M_s/s(Q), \mathbb{R}) \cong \mathfrak{h}^\vee$ with respect to the Killing form (see Donaldson's original work [Don16] and the recent work of Li [Li18] for more on maximal submanifolds). In our setup, we would like to reformulate this as a condition on the section s determining the flat spectral cover.

2. **Flat collapse:** As we have explained, the sLag torus fibration $T^*\mathcal{G}_6 \to \mathbb{R}^3/\mathbb{K}$ is singular over the $\mathbf{Y}$-vertex. The singular fibers are given by dicosms $\mathcal{G}_2$ over the three positive rays of the $\mathbf{Y}$ and a Hantzsche-Wendt space $\mathcal{G}_6$ over the origin. We have argued that the results of [LYZ04] imply that the total space admit a semi-flat Calabi-Yau metric. However, this is just a first approximation to the correct dual to M - presumably, instanton corrections should deform this geometry. Now, recent work of Bettiol, Derdzinski and Piccione [BDP17] classifies all flat deformations of the platycosms. These spaces can undergo quite interesting Gromov-Hausdorff collapse.

   One can raise the question whether instanton corrections to the semi-flat metric can induce collapse on the $\mathbf{Y}$-fibers. Although the generic torus fibers can only undergo rather trivial collapses, the other platycosms admit interesting features. The following example suggests that such collapse can contain nontrivial information on the quantum nature of the IIA/heterotic duality:

   Consider $\mathcal{G}_6$ seen as the central fiber in the above. The moduli space of flat deformations of $\mathcal{G}_6$ is $\cong \mathbb{R}^3_+$, and collapsed limits correspond to collapses of the isotypic components of the orthogonal representation of M_∇. For $\mathcal{G}_6$, $M_\nabla = \mathbb{K}$ and there are three isotypic components. As explained in [BDP17], collapse in each of the three directions produces a flat 2-orbifold $\mathbb{RP}^2(2, 2;)$, which is a $\mathbb{Z}_2$-quotient of $\mathbf{T}^2/\mathbb{Z}_2$, the $SU(2)$-character variety of a two-torus. This seems to suggest that the dual heterotic picture is corrected to include a singular $\mathbf{T}^2/\mathbb{Z}_2$-fiber. Meanwhile, the remaining exceptional fibers over the $\mathbf{Y}$-vertex are $\mathcal{G}_2$'s, which can collapse to either a $\mathbf{S}^1$, a Klein bottle or another $\mathbb{Z}_2$-quotient of $\mathbf{T}^2/\mathbb{Z}_2$, the flat 2-orbifold known as the *half pillowcase*, usually denoted $D^2(2, 2;)$.

3. **Kovalev-Lefschetz fibrations factoring through ramified covers:** A theorem of Hilden-Montesinos [Mon74] says that every closed orientable 3-manifold is a 3-fold branched cover over $\mathbf{S}^3$ with branched set over a knot K. The proof gives an explicit choice of K for a given 3-manifold Q. One could use this to relate Kovalev-Lefschetz fibrations over a general Q to *ramified Kovalev-Lefschetz fibrations* over $\mathbf{S}^3$. This should allow one to study G_2-flops for general Q using the ideas of [AV01] on flops for $(\mathbf{S}^3, K)$.

   One can also easily determine the knots associated to the platycosms. For example, working with 3-fold covers, $K(\mathcal{G}_6)$ is the figure-eight knot [Zim90]. If one would rather work with 2-fold covers, as in Donaldson's theory of branched harmonic maps [Don18], then the correct link to use for $\mathcal{G}_6$ are the Borromean rings.

4. **The role of gerbes/B-fields:** The SYZ Mirror Symmetry picture we described was under the condition of zero B-field. One might wonder how is the situation modified in the presence of a B-field $B \neq 0$, and furthermore, what is the extra structure on the G_2-side giving rise to B.

   There is a nice geometric answer in our context: such B-fields are geometrically $H/t(H)$-gerbes on the Calabi-Yau geometries, where $t(H)$ is the translation subgroup of H. In the G_2-geometry, the extra structure is an isogeny of tori $\widetilde{\mathbf{T}} \cong \mathbf{T}/t(H)$. This suggests that from the point of view of M-theory, the B-field is simply an artifact of the freedom in choosing finite subgroups of $SO(3)$ that are not Bieberbach groups, but extensions of such groups by affine translations.

5. **Codimension 7 singularities from degenerate spectral covers:** An important open problem in G_2-geometry is building manifolds of holonomy G_2 with point-like singularities beyond the conically singular cases. The correspondence between integrable G_2-structures and flat spectral covers suggest a new approach to this problem. In analogy with Hitchin systems on curves, one should expect a spectral correspondence relating flat Higgs bundles with non-generic Higgs fields to spectral covers ramifying over a link in Q. If one takes $\mathbf{S}^3$ and a link $L \subset \mathbf{S}^3$ such that $Q \to (\mathbf{S}^3, L)$ is a n-sheeted branched cover, one can interpret Q as part of ramified flat spectral data for a flat Higgs

bundle on $(\mathbf{S}^3, L)$, non-generic over L. E.g., one can take $Q = \mathcal{G}_6$ and L the Borromean rings, in which case $n = 2$. Over each strand one has a regular $SU(2)$-Higgs field (i.e., a 2×2-matrix which is a single Jordan block). One can take a limit in which L degenerates to a bouquet $\mathbf{S}^1 \sqcup \mathbf{S}^1 \sqcup \mathbf{S}^1$. In that situation, the Higgs field collapses over the central point to the 0-matrix (it becomes irregular - a multiple of the identity - and the only possible eigenvalue in $\mathfrak{su}(2)$ is 0). So over the singular point we have an element of the nilpotent cone. However, the associated G_2-geometry does not come from a smoothing of $\mathbb{C}^2/\mathbb{Z}_2$.

Bibliography

[Ach01] B. Acharya, *Confining Strings from G_2-holonomy spacetimes*, arXiv:hep-th/0101206.

[Ach98] B. Acharya, *M theory, Joyce Orbifolds and Super Yang-Mills*, Adv. Theor. Math. Phys. 3 (1999) 227-248.

[Ach00] B. Acharya, *On realising* N= 1 *super Yang-Mills in* M *theory*, arXiv:hep-th/0011089.

[AV01] M. Aganagic, C. Vafa, *Mirror symmetry and a G_2-flop*, JHEP 0305 (2003) 061.

[AP98] D. Arinkin, A. Polishchuk, *Fukaya category and Fourier transform*, arXiv:math/9811023 [math.AG]

[AMV01] M. Atiyah, J. Maldacena, C. Vafa, *An M-theory flop as a large N duality*, Journal of Mathematical Physics 42, 3209 (2001).

[AW01] M. Atiyah, E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, Advances in Theoretical and Mathematical Physics, 6 (1) (2001) pp. 1-106.

[AL14] R. Aurich, S. Lustig, *The Hantzsche-Wendt Manifold in Cosmic Topology*, Class. Quantum Grav. 31 (2014) 165009.

[BCHLTZ18] R. Barbosa, M. Čvetic, J. Heckman, C. Lawrie, E. Torres, G. Zoccarato, to appear.

[Bec18] F. Beck, *Aspects of Calabi-Yau integrable and Hitchin systems*, SIGMA 15 (2019), 001.

[BDP17] R. Bettiol, A. Derdzinski, P. Piccione, *Teichmüller theory and collapse of flat manifolds*, Ann. Mat. Pura Appl. (4) 197 (2018), no. 4, 12471268.

[BL99] C. Birkenhake, H. Lange, *Complex Tori*, Birkhäuser Basel (1999).

[Blu11] M. Blume, *Construction of G-Hilbert schemes*, Math. Nachr. 284 (2011), 953-959.

[BFM02] A. Borel, R. Friedman, J. Morgan, *Almost commuting elements in compact Lie groups*, Mem. Amer. Math. Soc. 157 (2002), no. 747.

[BSN17] A. Braun, S. Schäfer-Nameki, *Compact, singular G_2-holonomy manifolds and M/heterotic/F-theory duality*, J. High Energ. Phys. (2018) 2018: 126.

[BCHSN18] A. Braun, S. Cizel, M. Hubner, S. Schäfer-Nameki, *Higgs bundles for M-theory on G_2-manifolds*, arXiv:1812.06072 [hep-th]

[Bri07] T. Bridgeland, *Stability conditions on triangulated categories*, Annals of Mathematics, 166 (2007), 317345.

[Bri68] E. Brieskorn, *Die Auflosung der rationalen Singularitaten holomorphen Abbildungen*, Math. Ann. 178 (1968), 255270.

[Cha86] L. Charlap, *Bieberbach groups and flat manifolds*, Springer-Verlag New York (1986).

[CC16] G. Chen, X. Chen, *Gravitational instantons with faster than quadratic curvature decay (III)*, arXiv:1603.08465 [math.DG]

[CR03] J. Conway, J. Rossetti, *Describing the platycosms*, arXiv:math/0311476 [math.DG].

[Cor88] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. 28 (1988), no. 3.

[Cos13] K. Costello, *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*, Pure and Applied Mathematics Quarterly Volume 9, Number 1 (Special Issue: In honor of Dennis Sullivan, Part 1 of 2) 73165 (2013).

[Cul86] M. Culler, *Lifting representations to covering groups*, Advances in Mathematics 59, 64-70 (1986).

[Deb99] O. Debarre, *Tores et variétés abéliennes complexes*, Cours Spécialisés, 6. Société Mathématique de France, Paris; EDP Sciences, Les Ulis (1999).

[DDP07] D.-E. Diaconescu, R. Donagi, T. Pantev, *Geometric transitions and mixed Hodge structures*, Adv. Theor. Math. Phys. 11 (2007), no. 1, 6589.

[DDP06] D.-E. Diaconescu, R. Donagi, T. Pantev, *Intermediate Jacobians and ADE Hitchin systems*, Math. Res. Lett. 14 (2007), no. 5, 745756.

[Don95] R. Donagi, *Spectral covers*, In: Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), volume 28 of Math. Sci. Res. Inst. Publ., pages 6586. Cambridge Univ. Press, Cambridge (1995).

[DM95] R. Donagi, E. Markman, *Spectral curves, algebraically completely integrable Hamiltonian systems, and moduli of bundles*, In: Francaviglia M., Greco S. (eds) Integrable Systems and Quantum Groups. Lecture Notes in Mathematics, vol. 1620. Springer, Berlin, Heidelberg (1995).

[Don16] S. Donaldson, *Adiabatic limits of co-associative Kovalev-Lefschetz fibrations*, Algebra, geometry, and physics in the 21st century, 129, Progr. Math., 324, Birkhuser/Springer, Cham, (2017).

[Don18] S. Donaldson, *Adiabatic limits, multi-valued harmonic functions and the Nash-Moser-Zehnder theory*, Talk at Imperial College, London, June 2018.

[Don87] S. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) 55 (1987).

[GW11] D. Gaiotto, E. Witten, *Knot Invariants from Four-Dimensional Gauge Theory*, Adv. Theor. Math. Phys. 16 (2012), no. 3, 9351086.

[Gol88] W. Goldman, *Geometric structures on manifolds and varieties of representations*, Geometry of group representations (Boulder, CO, 1987), 169198, Contemp. Math., 74, Amer. Math. Soc., Providence, RI (1988).

[GY08] S. Grigorian, S.-T. Yau, *Local geometry of the G_2 moduli space*, Comm. Math. Phys. 287 (2009), no. 2, 459488.

[Gro98] M. Gross, *Special Lagrangian fibrations. I. Topology*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ (1998) pp. 156193.

[Gro99] M. Gross, *Special Lagrangian fibrations. II. Geometry*, A survey of techniques in the study of special Lagrangian fibrations, Surveys in differential geometry: differential geometry inspired by string theory, Surv. Differ. Geom., vol. 5, Int. Press, Boston, MA (1999) pp. 341403.

[Hit87a] N. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. 54, 1 (1987) 91-114.

[Hit00] N. Hitchin, *The Geometry of Three-Forms in Six Dimensions*, J.Diff.Geom. 55 (2000) no.3, 547-576.

[Hit87] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) 55 (1987) 59126.

[HKLR87] N. Hitchin, A. Karlhede, U. Lindström, M. Roček, *Hyper-Kähler metrics and supersymmetry*, Comm. Math. Phys. Volume 108, Number 4 (1987), 535-589.

[Joy00] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press (2000).

[Joy07] D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford University Press (2007).

[KS99] V. Kac, A. Smilga, *Vacuum structure in supersymmetric Yang-Mills theories with any gauge group*, The many faces of the superworld, 185234, World Sci. Publ., River Edge, NJ (2000).

[KW06] A. Kapustin, E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, Commun. Number Theory Phys. 1 (2007), no. 1, 1236.

[KL07] S. Karigiannis, N. Leung, *Hodge Theory for G_2-manifolds: Intermediate Jacobians and Abel-Jacobi maps*, Proc. Lond. Math. Soc. (3) 99 (2009), no. 2, 297325.

[KN79] G. Kempf, L. Ness, *The length of vectors in representation spaces*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., 732, Berlin, New York: Springer-Verlag, pp. 233243

[KN63] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Vol. 1*, Wiley Classics Library (1996).

[Kro89] P. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. Volume 29, Number 3 (1989), 685-697.

[Kro89a] P. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. Volume 29, Number 3 (1989), 665-683.

[Leu01] N. Leung, *Mirror Symmetry without corrections*, Comm. Anal. Geom. 13 (2005), no. 2, 287331.

[LYZ02] N. Leung, S.-T. Yau, E. Zaslow, *From special lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv.Theor.Math.Phys. 4 (2002) 1319-1341.

[Li18] Y. Li, *Dirichlet problem for maximal graphs of higher codimension*, arXiv:1807.11795 [math.DG]

[LYZ04] J. Loftin, S.-T. Yau, E. Zaslow, *Affine Manifolds, SYZ Geometry, and the "Y" Vertex*, J. Differential Geom. 71 (2005), no. 1, 129158.

[Loo84] E. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Series 77, Cambridge University Press (1984).

[Mon74] J. Montesinos, *A Representation of closed, orientable 3-manifolds as 3-fold branched coverings of* S^3, Bull. Amer. Math. Soc. 80 (1974), 845846.

[Mor02] D. Morrison, *Half* $K3$ *surfaces*, talk at Strings 2002, Cambridge University, July 2002.

[PTVV11] T. Pantev, B. Toen, M. Vaquie, G. Vezzosi, *Shifted symplectic structures*, arXiv:1111.3209 [math.AG]

[PW11] T. Pantev, M. Wijnholt, *Hitchin's Equations and M-Theory Phenomenology*, J. Geom. Phys. 61 (2011), no. 7, 12231247.

[Pfä00] F. Pfäffle, *The Dirac spectrum of Bieberbach manifolds*, J. Geom. Phys. 35 (4) (2000), 367-385.

[Sch12] L. Schaposnik, *Spectral data for G-Higgs bundles*, Oxford University D.Phil. Thesis (2012).

[Sco83] P. Scott, *The Geometries of 3-manifolds*, Bull. London Math. Soc, 15 (1983), 401-487.

[Scz12] A. Sczcepański, *Geometry of crystallographic groups*, Vol. 4, Series Algebra and Discrete Mathematics, World Scientific (2012).

[Sen97] A. Sen, *A Note on Enhanced Gauge Symmetries in M- and String Theory*, JHEP 9709:001 (1997) .

[Sim92] C. Simpson, *Higgs bundles and local systems*, Publ. Math. I. H. E. S. 75 (1992) 595.

[Slo80] P. Slodowy, *Simple Singularities and Simple Algebraic Groups*, Springer-Verlag Berlin Heidelberg (1980).

[Sze04] B. Szendrői, *Artin group actions on derived categories of threefolds*, J. reine angew. Math. 572 (2004), 139166.

[SYZ96] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror Symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243259.

[Tah71] K. Tahara, *On the Finite Subgroups of* $GL(3, \mathbb{Z})$, Nagoya Math. J. 41 (1971) 169209.

[TY90] G. Tian, S.-T. Yau, *Complete Kahler manifolds with zero Ricci curvature*, I. J. Amer. Math. Soc. 3 (1990), no.3, 579-609.

[Tju70] G. Tjurina, *Resolution of singularities of plane deformations of double rational points*, Funct. Anal. Appl., 4:1 (1970), 6873.

[Zim86] B. Zimmerman, *Finite group actions on Haken* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) 37 (1986), no. 148, 499511.

[Zim90] B. Zimmerman, *On the Hantzsche-Wendt manifold*, Mh. Math. 110, 321-327 (1990).

[Wit96] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. B460 (1996).

[Wit12] E. Witten, *Fivebranes and knots*, Quantum Topol. 3 (2012), no. 1, 1137.

[Wit95] E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B 443, 85 (1995).