Preliminary Examination, Part I

Monday morning, May 9, 2005

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Is there a non-abelian group of order $n = 49$? Either find one or explain why none exists. Do the same for $n = 50$ and $n = 51$.

2. The number e is given by the series

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + \cdots$$

Prove $e < 3$.

3. Let A be any real (3×3)-matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

a) Show $X \cdot AX = 0$ for all $X \in \mathbb{R}^3$ (where $X \cdot Y$ is the usual dot product).

b) Find a non-zero vector Y so that $AY = 0$.

c) For Y as in part (b), show that $AX \cdot Y = 0$ for all $X \in \mathbb{R}^3$.

d) For Y as in part (b), show there is a real number λ so that if X is any vector orthogonal to Y (i.e., $X \cdot Y = 0$) then $A^2 X = \lambda X$. Determine λ.

Justify your assertions.

4. Compute the following limits if they exist and justify your conclusion.

a) $\lim_{\lambda \to +\infty} \int_0^1 \cos(\lambda x) dx$

b) $\lim_{\lambda \to +\infty} \int_0^1 |\cos(\lambda x)| dx$.

5. a) Let $G = GL(n, \mathbb{R})$ and $H = \{A \in GL(n, \mathbb{R}) | \det A > 0\}$ where $n > 1$. Is H a subgroup of G? If so, is it a normal subgroup?

b) Do the same with H replaced by $\{A \in GL(n, \mathbb{R}) | AA^t = I\}$ where A^t denotes the transpose of A.

c) Do the same with H replaced by $\{A \in GL(n, \mathbb{R}) | A = A^t\}$ where A^t denotes the transpose of A.

6. a) Let X be a metric space, and let $x_1, x_2, x_3, \dots \in X$. Prove that the sequence $\{x_n\}$ can have at most one limit $\lim_{n\to\infty} x_n$ in X.

   b) Does the same conclusion hold if X is just assumed to be an arbitrary topological space? Explain.

Preliminary Examination, Part II

Monday afternoon, May 9, 2005

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7. Let $\vec{F}$ be the vector field

$$\vec{F} = (x^3, y^3, z^3) \ , \quad \text{i.e.} \qquad \vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

Compute the surface integral

$$\iint_S \vec{F} \cdot \vec{n} \ dS$$

over the surface of a unit sphere.

8. Suppose f is a C^∞-function (f has derivatives of all orders) on the closed interval $[0, 3]$ and

$$f(0) = 0, \qquad f(1) = 1, \qquad f(2) = -1, \qquad f(3) = 0 \ .$$

Prove the second derivative $f''(x)$ has at least one zero in the open interval $(0, 3)$.

9. Which of the following rings is an integral domain? Which is a field? Justify your assertions.

a) $\mathbb{Z}[x]/(x^2 + 7)$

b) $\mathbb{R}[x]/(x^4 + 3x^2 + 2)$

c) $\mathbb{Q}[x]/(x^3 - 2)$

10. A certain smooth connected curve C in the plane intersects all the curves of the form $xy = k$ (for $k \in \mathbb{R}$) at right angles. If C passes through the point $(1, 1)$, where does C meet the line $x = 2$?

11. Suppose A is a real symmetric $(n \times n)$-matrix with eigenvalues $1, 2, \cdots, n-1, n$. Compute $\| A \|$ the norm of A where

$$\| A \| = \sup\{\| A\vec{x} \| \text{ for all vectors } x \in \mathbb{R}^n \text{ with norm } \| \vec{x} \| = 1\} \ ,$$

where $\| \vec{x} \|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ for $\vec{x} = (x_1, \cdots, x_n)$. Justify your conclusion.

12. a) Suppose $f(x)$ is a continuous real valued function, for $x \in [0, \infty)$, and

$$\lim_{x \to \infty} f(x) = 1 \ .$$

Prove f is uniformly continuous for $x \in [0, \infty)$.

b) Give an example of a function g that is uniformly continuous on $[0, \infty)$, such that $\lim_{x \to \infty} g(x)$ does not exist. (You don't have to prove this, just give an example.)