This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.
1. Is there a non-abelian group of order $n = 49$? Either find one or explain why none exists. Do the same for $n = 50$ and $n = 51$.

2. The number $e$ is given by the series

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + \cdots$$

Prove $e < 3$.

3. Let $A$ be any real $(3 \times 3)$-matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$  

a) Show $X \cdot AX = 0$ for all $X \in \mathbb{R}^3$ (where $X \cdot Y$ is the usual dot product).

b) Find a non-zero vector $Y$ so that $AY = 0$.

c) For $Y$ as in part (b), show that $AX \cdot Y = 0$ for all $X \in \mathbb{R}^3$.

d) For $Y$ as in part (b), show there is a real number $\lambda$ so that if $X$ is any vector orthogonal to $Y$ (i.e., $X \cdot Y = 0$) then $A^2X = \lambda X$. Determine $\lambda$.

Justify your assertions.

4. Compute the following limits if they exist and justify your conclusion.

a) $\lim_{\lambda \to +\infty} \int_0^1 \cos(\lambda x)dx$

b) $\lim_{\lambda \to +\infty} \int_0^1 |\cos(\lambda x)|dx$.

5. a) Let $G = GL(n, \mathbb{R})$ and $H = \{ A \in GL(n, \mathbb{R}) | \det A > 0 \}$ where $n > 1$. Is $H$ a subgroup of $G$? If so, is it a normal subgroup?

b) Do the same with $H$ replaced by $\{ A \in GL(n, \mathbb{R}) | AA^t = I \}$ where $A^t$ denotes the transpose of $A$.

c) Do the same with $H$ replaced by $\{ A \in GL(n, \mathbb{R}) | A = A^t \}$ where $A^t$ denotes the transpose of $A$. 

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6. a) Let $X$ be a metric space, and let $x_1, x_2, x_3, \cdots \in X$. Prove that the sequence $\{x_n\}$ can have at most one limit $\lim_{n \to \infty} x_n$ in $X$.

b) Does the same conclusion hold if $X$ is just assumed to be an arbitrary topological space? Explain.
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7. Let \( \vec{F} \) be the vector field
\[
\vec{F} = (x^3, y^3, z^3), \quad \text{i.e.} \quad \vec{F} = x^3 \vec{i} + y^3 \vec{j} + z^3 \vec{k}
\]
Compute the surface integral
\[
\iint_S \vec{F} \cdot \vec{n} \, dS
\]
on the surface of a unit sphere.

8. Suppose \( f \) is a \( C^\infty \)-function (\( f \) has derivatives of all orders) on the closed interval \([0, 3]\) and
\[
f(0) = 0, \quad f(1) = 1, \quad f(2) = -1, \quad f(3) = 0.
\]
Prove the second derivative \( f''(x) \) has at least one zero in the open interval \((0, 3)\).

9. Which of the following rings is an integral domain? Which is a field? Justify your assertions.
   a) \( \mathbb{Z}[x]/(x^2 + 7) \)
   b) \( \mathbb{R}[x]/(x^4 + 3x^2 + 2) \)
   c) \( \mathbb{Q}[x]/(x^3 - 2) \)

10. A certain smooth connected curve \( C \) in the plane intersects all the curves of the form \( xy = k \) (for \( k \in \mathbb{R} \)) at right angles. If \( C \) passes through the point \((1, 1)\), where does \( C \) meet the line \( x = 2? \)

11. Suppose \( A \) is a real symmetric \((n \times n)\)-matrix with eigenvalues \( 1, 2, \cdots, n-1, n \). Compute \( \| A \| \) the norm of \( A \) where
\[
\| A \| = \sup \{ \| A\vec{x} \| \ \text{for all vectors} \ \vec{x} \in \mathbb{R}^n \ \text{with norm} \ \| \vec{x} \| = 1 \},
\]
where \( \| \vec{x} \|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \) for \( \vec{x} = (x_1, \cdots, x_n) \). Justify your conclusion.
12. a) Suppose $f(x)$ is a continuous real valued function, for $x \in [0, \infty)$, and
\[
\lim_{x \to \infty} f(x) = 1 .
\]
Prove $f$ is uniformly continuous for $x \in [0, \infty)$.

b) Give an example of a function $g$ that is uniformly continuous on $[0, \infty)$, such that $\lim_{x \to \infty} g(x)$ does not exist. (You don’t have to prove this, just give an example.)