Preliminary Examination, Part I

Monday morning, April 24, 2006

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation:

$$T : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 9 \\ 3 & 6 & 8 & 8 \\ 0 & 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

What are dim (ker T) and dim (im T)? Give a basis for ker (T) (Note: ker (T) is the null space of T).

2. Let $f(x)$ be the rational function

$$f(x) = \frac{x}{1 + x^2}$$

a) Compute the value of the 100^{th} derivative at $x = 0$

b) Compute the value of the 100^{th} derivative at $x = 1$.

3. A square $n \times n$ matrix M is *diagonalized* by an invertible matrix P if PMP^{-1} is a diagonal matrix. Of the following three matrices, one can be diagonalized by an orthogonal matrix, one can be diagonalized but not by any orthogonal matrix, and one cannot be diagonalized. State which is which and why.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}$$

4. Let $\{\phi_n\}$ be a sequence of continuous, real valued functions defined on $\mathbb{R}$ such that

$$\phi_n(x) = 0 \qquad \text{for all} \quad |x| \geq \frac{1}{n},$$

$$\phi_n(x) \geq 0 \qquad \text{for all} \quad x \in \mathbb{R}, \text{and}$$

$$\int_{-1}^{1} \phi_n(x) dx = 1.$$

For each continuous function $f : \mathbb{R} \to \mathbb{R}$, define the sequence $\{f_n\}$ by

$$f_n(x) = \int_{-\infty}^{\infty} \phi_n(x - y) f(y) dy.$$

a) Prove that the sequence $\{f_n(x)\}$ converges to $f(x)$ for every $x \in \mathbb{R}$.

b) If $f(x) = 0$, for $|x| \geq 10$, prove that the functions f_n converge uniformly to f.

5. Exhibit two non-abelian groups of order 18. In other words, exhibit two non-commutative groups H and G with 18 elements, and show that H and G are not isomorphic.

6. Let A be a subset of the plane $\mathbb{R}^2$, with the usual euclidian metric d. Prove that the following statements are equivalent:

a) A is closed.

b) Every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements $x_n \in A$ such that

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty,$$

converges to a limit in A.

Preliminary Examination, Part II
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7. Evaluate the line integral

$$\oint_C (x^2 e^x + y\cos(x))dx + (\sin(x) + x + \cos(y))dy$$

where C is the trapezoid with vertices at $(-3, 0)$, $(2, 0)$, $(1, 3)$, $(-2, 3)$, oriented counterclockwise.

8. Suppose u is a twice differentiable function on $\mathbb{R}$ which satisfies the differential equation

$$\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} - c(x)u = 0,$$

where $b(x)$ and $c(x)$ are continuous functions on $\mathbb{R}$ with $c(x) > 0$ for every $x \in (0, 1)$.

a) Show that u cannot have a positive local maximum in the interval $(0, 1)$. Also show that u cannot have a negative local minimum in $(0, 1)$.

b) If $u(0) = u(1) = 0$, prove that $u(x) = 0$ for every $x \in [0, 1]$.

9. For each of the following equations, either find a solution in integers or explain why none exists.

a) $7x + 21y = 43$

b) $7x + 22y = 43$

c) $x^2 + y^2 = 4003$

10. a) Evaluate the double integral:

$$\iint_{\mathbb{R}^2} \frac{dx\,dy}{[1 + 4x^2 + 9y^2]^2}\ .$$

b) Evaluate the double integral:

$$\iint_{\mathbb{R}^2} \frac{du\,dv}{[1 + 4(u + 3v)^2 + 9(u - v)^2]^2}\ .$$

11. Which of the following commutative rings are fields? Explain

$\mathbb{R}[x]/(x^2 + x + 1)$; $\mathbb{C}[x]/(x^2 + x + 1)$; $\mathbb{F}_4[x]/(x^2 + x + 1)$,

where $\mathbb{F}_4$ is the field of four elements.

12. Let A be the matrix

$$A = \begin{bmatrix} -1 & 1 \\ c & -1 \end{bmatrix}$$

where $c \in \mathbb{R}$ is a real number. Find all the values of c for which every solution of the differential equation

$$\frac{dV}{dt} = AV$$

satisfies

$$\lim_{t \to \infty} V(t) = 0.$$

Here $V : \mathbb{R} \to \mathbb{R}^2$ is a vector-valued function on $\mathbb{R}$, which can be viewed as a column vector whose two components are real-valued functions on $\mathbb{R}$.