Preliminary Examination, Part I

Monday morning, April 23, 2007

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 1, 1, 1), (3, 1, 3, 1), (3, 2, 2, -1)$.

2. a) Let f be a real-valued function defined on the closed interval $[0, 1]$. Give a careful definition of the statement that the Riemann integral $\int_0^1 f(x)dx$ is equal to a real number c .

   b) Prove that if the Riemann integral in (a) exists, then the function f is bounded on $[0, 1]$.

3. Let G be a finite group. Suppose for every $g \in G$ other than the identity element e, there is a subgroup $H \subset G$ of index 2 that does not contain g.

   a) Show that $g^2 = e$ for all $g \in G$.

   b) Show that G is abelian.

4. Find the volume of the ellipsoid defined by the inequality

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \leq 1 \ .$$

5. Let A, B be $n \times n$ real matrices.

   a) Show that if B is invertible, then the matrices A and BAB^{-1} have the same eigenvalues. Find the relationship between the eigenvectors of these two matrices.

   b) Show that if c is an eigenvalue of AB, then c is also an eigenvalue of BA. [Note: B need not be invertible in this part.]

6. Evaluate the line integral $\int_C (x - y)dx + (x + y)dy$ where C is the quarter circle $x^2 + y^2 = 25$, $x \geq 0$, $y \geq 0$, oriented from $(5, 0)$ to $(0, 5)$.

PRELIMINARY EXAMINATION, PART II

Monday afternoon, April 23, 2007

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for all $x \in \mathbb{R}$, $|f(x) - f(f(x))| \leq \frac{1}{2}|f(x) - x|$. Prove that there is an $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

8. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers.
   a) Show that if $\Sigma_{n=1}^{\infty}|a_n|$ converges, then $\Sigma_{i=1}^{\infty} a_{n_i}$ converges for every increasing sequence of positive integers $n_1 < n_2 < \ldots$.
   b) Does the converse of (a) hold? Give either a proof or a counterexample.

9. a) Suppose that A is a 2×2 real matrix whose determinant is 6 and whose trace is 5. Prove that A is diagonalizable, and find a diagonal matrix that is similar to A.
   b) Does the conclusion still hold if instead $\det A = 9$ and $\operatorname{tr} A = 6$? Justify your assertion.

10. Define $f(x) = x \sin(1/x)$ for $x \neq 0$, and define $f(0) = 0$.
    a) Prove that f is a continuous function on the real line.
    b) Find all $x \in \mathbb{R}$ such that f is differentiable at x. Prove your assertion.

11. Let p be a prime number, let $F = \mathbb{Z}/p\mathbb{Z}$, and let $f(t) \in F[t]$ be an irreducible polynomial of degree d.
    a) Prove that $f(t)$ divides $t^{p^d} - t$. [Hint: Consider $F[t]/(f)$.]
    b) Prove more generally that $f(t)$ divides $t^{p^n} - t$ if and only if d divides n.

    [Note: a^{b^c} means $a^{(b^c)}$.]

12. a) Give an example of a real-valued continuous function f on the open interval $0 < x < 1$ that is not *uniformly* continuous on this interval. Give a brief explanation of why your example works.
    b) For your f, prove that if $0 < a < b < 1$ then f is uniformly continuous on the open interval $a < x < b$.