PRELIMINARY EXAMINATION, PART I

Thursday morning, April 28, 2010

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Let $a < b$ be real numbers and $\{x_n\}$ be a sequence of real numbers. Show that if $\{x_n\}$ is a monotonically increasing sequence of points in $(a, b]$, then $\{x_n\}$ converges to a point of $(a, b]$. Give a sequence showing that the converse of this statement is false.

2. Two matrices A and A' in $\mathrm{Mat}_2(\mathbb{Q})$ are similar if $A' = CAC^{-1}$ for some invertible matrix $C \in \mathrm{GL}_2(\mathbb{Q})$. In this case, say A and A' are in the same similarity class. Find representatives for all similarity classes of matrices A in $\mathrm{Mat}_2(\mathbb{Q})$ which have multiplicative order 6 in the sense that $n = 6$ is the smallest positive integer such that A^n is the identity matrix.

3. Let $\mathcal{C}(0, 1)$ be the space of continuous functions on the interval $[0, 1]$ and define

$$||f||_1 = \int_0^1 |f(x)|dx$$
$$||f||_0 = \int_0^1 x|f(x)|dx.$$

Assume that $|| \cdot ||_1$ is a norm on $\mathcal{C}(0, 1)$ and show that $|| \cdot ||_0$ is a norm on $\mathcal{C}(0, 1)$. Is $|| \cdot ||_1$ equivalent to $|| \cdot ||_0$?

4. Prove that $\sqrt{2} + \sqrt{3}$ is not a rational number.

5. Let X be a metric space. Either prove or give a counterexample for the following statements:

   (a) If X is compact, it is complete.

   (b) If X is complete, it is compact.

6. Let R be the region in the first quadrant of (x, y)-plane that bounded by the y-axis and the curves $y = \sin(x)$ and $y = \cos(x)$. Let C be the boundary of R oriented counterclockwise. Compute the line integral

$$\int_C (y + \ln(y + 1))dx + \frac{x + e^y}{y + 1}dy$$

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7. Let (X, d) be a metric space, let $a \in X$, $r > 0$, and let $B(a, r)$ denote the open ball around a of radius r. In other words

$$B(a, r) = \{x \in X : d(x, a) < r\}$$

   (a) Show that

   $$\overline{B(a, r)} \subset \{x \in X : d(x, a) \leq r\}$$

   where $\overline{B(a, r)}$ is the closure of $B(a, r)$.

   (b) Can the inclusion in part (a) be proper? Give either an example where the inclusion is proper, or prove the sets are always equal.

8. How many isomorphism classes of non-abelian finite groups G of order 20 are there? Justify your answer.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let a be a nonzero real number. Show that the function

$$F(x) := \frac{1}{2a} \int_{-a}^{a} f(x + t) dt$$

is differentiable and has a continuous derivative.

10. Let F be a finite field. Show that any function from F to F is a polynomial function.

11. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : [0, 1] \to [0, 1]$ which is non-increasing with n, in the sense that $f_{n+1}(x) \leq f_n(x)$ for all $n \geq 1$ and all $x \in [0, 1]$. Suppose that for each $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) = 0$. Show that the $f_n(x)$ converge uniformly to 0 for $x \in [0, 1]$.

12. Let R be a commutative ring with multiplicative identity 1_R different from the additive identity 0_R.

   (a) Show that if R is an integral domain with finitely many elements, then R must be a field.

   (b) Show that if R has 10 elements, then R is isomorphic to the ring $\mathbb{Z}/(\mathbb{Z} \cdot 10)$.