Preliminary Examination, Part I

Wednesday morning, April 27, 2011

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Justify the assertions you make. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+y) = f(x) + f(y)$ for all x, y. Suppose that there exist $a, M > 0$ such that $|f(x)| \le M$ for all $x \in [-a, a]$. Prove that f is uniformly continuous.

2. For each of the following either give an example or explain why no such example exists.

a) An abelian group of order 110 that is not cyclic.

b) A group of order 14 that is not abelian.

c) A group of order 33 that is not cyclic.

3. Let X be a metric space, and let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of non-empty subsets of X whose union is X. Let $f : X \to \mathbb{R}$ be a function such that the restriction of f to A_i is continuous for each i.

a) Show that if each A_i is open then f is continuous.

b) Give an example to show that f need not be continuous if instead each A_i is closed.

4. Let x be any positive real number, and define a sequence $\{a_n\}$ by

$$a_n = \frac{[x] + [2x] + [3x] + \cdots + [nx]}{n^2}$$

where $[x]$ is the largest integer less than or equal to x. Prove that $\lim_{n\to\infty} a_n = x/2$.

5. Consider the following inner product on $\mathbb{R}^3$:

$$\langle (a,b,c), (d,e,f) \rangle = (a,b,c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix}.$$

Use the Gram-Schmidt process, starting with the basis $\{(1,0,0), (0,1,0), (0,0,1)\}$, to produce an orthonormal basis of this inner-product space.

6. Consider the nonlinear ordinary differential equation

$$u'' - u^2 + 9 = 0.$$

a) Convert the equation to a system of first order equations and sketch some representative solutions of this first order system. Identify all constant solutions in your sketch.

b) Find a vector field which is orthogonal to the trajectories of your first order system.

c) Find a scalar function on the plane whose gradient is the vector field you found in (b).

Preliminary Examination, Part II

Wednesday afternoon, April 27, 2011

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7. Assume that f is an infinitely differentiable real-valued function defined on the real line, with $f(1) = 1$, $f(2) = 2$, and $f(3) = 0$. Must there exist an open interval (a, b) contained in the closed interval $[1, 3]$ on which the graph of f is concave down? Give either a proof or a counterexample.

8. Consider the complex vector space V_1 consisting of polynomials of degree less than n, and the complex vector space $V_2 = \mathbb{C}[x]/(x^n)$. Consider the linear operators $D, L : V_1 \to V_1$ given by $D(f(x)) = f'(x)$ (differentiation) and $L(f) = xf'(x)$. Also consider the linear operator $M : V_2 \to V_2$ given by multiplication by x.

a) Are there bases B_1, B_2 of V_1, V_2 with respect to which D, M have the same matrix? If so, find such bases. If not, show why not.

b) Do the same for L, M.

9. Let $f : X \to Y$ be a surjective map between two metric spaces (X, d_X) and (Y, d_Y) such that

$$d_Y(f(x_1), f(x_2)) = d_1(x_1, x_2)$$

for all $x_1, x_1 \in X$. Prove that f defines a homeomorphism between X and Y.

10. Let c_1, c_2, c_3 be positive real numbers, let e_1, e_2, e_3 be the standard unit vectors in $\mathbb{R}^3$, and let $v_i = c_i e_i$ for $i = 1, 2, 3$. Let $v_0 = v_1 + v_2 + v_3$. Find the volume of the polyhedron with vertices v_0, v_1, v_2, v_3.

11. Let A be the ring of continuous functions $f : [0, 1] \to \mathbb{R}$ under pointwise addition and multiplication. Let I be the subset of A consisting of functions that vanish at 0.

a) Determine whether A is an integral domain.

b) Prove that I is a prime ideal of A, and determine if it is a maximal ideal.

12. Recall that for a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, the Laplacian of f is defined by

$$\Delta f = \operatorname{div}(\operatorname{grad} f).$$

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function satisfying $f(\vec{x}) = 1/\|\vec{x}\|$ for $\|\vec{x}\| \geq 1$.

a) Verify that $\Delta f(\vec{x}) = 0$ for $\|\vec{x}\| \geq 1$.

b) Compute $\int_{\mathbb{R}^3} \Delta f \, dV$.