Preliminary Examination, Part I

Wednesday morning, April 25, 2012

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name on each workbook you submit. All problems have equal weight.

1. (a) Let I denote the closed interval [-1,1] . Show that every continuous function $f : I \to I$ has a fixed point, i.e., a point $x \in I$ such that $f(x) = x$.

(b) Is the same statement true for the open interval $J = (-1, 1)$? In other words, does every continuous function $f : J \to J$ have a fixed point? Justify your answer.

2. Let $a_1, a_2, a_3 \ldots$ be a sequence of real numbers. Suppose that for every strictly increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ that *omits* infinitely many positive integers, the subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ converges. Must the sequence $a_1, a_2, a_3, \ldots$ converge? Either prove that it does or provide a counterexample.

3. Which of the following rings are commutative? Of those that are, which are integral domains? Which are principal ideal domains?

(a) The ring of 4×4 matrices over $\mathbf{R}$.

(b) $\mathbf{R} \times \mathbf{R}$, under componentwise addition and multiplication.

(c) $\mathbf{R}[x, y]$.

(d) $\mathbf{Q}[t]$.

4. Consider the differential equation

$$x^2y'' + xy' + (x^2 - 4)y = 0.$$

Find all solutions of this equation that are of the form $y(x) = \sum_{n=0}^{\infty} c_n x^{n+2}$.

5. Consider the function $f(x) = 1/x$ on $x > 0$.

(a) Is f uniformly continuous on the set $x > 1$?

(b) Is f uniformly continuous on the set $0 < x < 1$?

6. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} 2x & + & y & + & 6z & - & 7w & = & 0 \\ 2x & + & 2y & + & 8z & - & 9w & = & 0 \\ 2x & + & 3y & + & 10z & - & 11w & = & 0 \end{array}$$

Find an orthonormal basis for the solution space to this system, viewed as a subspace of $\mathbf{R}^4$,with the standard Euclidean inner product.

Preliminary Examination, Part II

Wednesday afternoon, April 25, 2012

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7. Prove or disprove each of the following:

(a) Every group of order 10 has a non-trivial proper normal subgroup.
(b) Every group of order 10 has a non-trivial center.
(c) Every group of order 10 is the direct product of its Sylow subgroups.

8. Consider the function $f(x) = x^5 - 2x^4 + 3x^3 + 2$.

(a) Prove that f is a bijection $\mathbf{R} \to \mathbf{R}$.
(b) At what points is f^{-1} continuous, and at what points is it differentiable?

9. Let $c \in \mathbf{R}$, and consider the matrix

$$A = \begin{pmatrix} 3-c & -c & 1 \\ c-1 & 2+c & -1 \\ c+1 & c & 3 \end{pmatrix}.$$

Find all values of c such that $\mathbf{R}^3$ has a basis consisting of eigenvectors of A. For each such c, find such a basis.

10. Determine the value of the line integral $\int_C \vec{F} \cdot d\vec{\ell}$, where

$$\vec{F} = (e^{-y} - ze^{-x}, e^{-z} - xe^{-y}, e^{-x} - ye^{-z})$$

and C is the path from $(0,0,0)$ to $(1,1,1)$ given by

$$x = \frac{1}{\log 2} \log(1+t)$$

$$y = sin(\frac{\pi t}{2})$$

$$z = \frac{1-e^t}{1-e}$$

11. Suppose X is a Hausdorff space. Show that if $A \subset X$ is compact and $x \in X$ is not in A, then there exist disjoint open sets U and V containing A and x respectively.

12. Let V be a vector space, let v be an element of V, and write $T : V \to V$ for a linear transformation. Suppose, for some $m \geq 0$, that $T^m(v) \neq 0$ and $T^{m+1}(v) = 0$. Prove that the vectors $v, T(v), \cdots, T^m(v)$ are linearly independent.