Preliminary Examination, Part I

Wednesday morning, May 8, 2013

This part of the examination consists of six problems. You should work all of the problems. Show all of your work in your workbook. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name on each workbook you submit. All problems have equal weight.

## Prelim Exam: Spring 2013
## Morning session

1. Find representatives for the conjugacy classes in $\mathrm{GL}_2(\mathbb{R})$ of all matrices $A$ such that $A^2 - A$ is the two-by-two identity matrix.

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with entries in $\mathbb{R}$. Show that all solutions of the system $Y' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y$ approach zero as $t \to \infty$ if and only if $tr(A) < 0$ and $det(A) > 0$.

3. Prove that a topological space $X$ is Hausdorff if and only if the diagonal
$$\Delta = \{(x, x) \in X \times X \mid x \in X\}$$
is closed in $X \times X$.

4. Suppose $G$ is a group of order $2n$ where $n$ is an odd integer.

   a. Show that there is an injective homomorphism from $G$ to a symmetric group $S_{2n}$ on $2n$ letters such that an element $h$ of order 2 in $G$ goes to the product of $n$ transpositions.

   b. Show that $G$ has a normal subgroup of order $n$.

5. What is the degree over the field $F = \mathbb{Z}/2$ of the splitting field of the polynomial $x^7 - 1$?

6. Evaluate
$$\int_C x^2 y \; dx + xy \; dy$$
counter-clockwise around the boundary $C$ of the region $R$ given by all $(x, y)$ for which
$$x^4 \le y \le -x^2 + 2.$$

Preliminary Examination, Part II

Wednesday afternoon, May 8, 2013

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**Prelim Exam: Spring 2013**
**Afternoon session**

7. Is the ring $\mathbb{Z}[x]$ a principal ideal domain? You should either prove that it is or prove that it is not.

8. Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = (2xy+4x, -y^2+e^{z^2}, \sin(xy^2))$, and let $S$ be the boundary of the region $R$ underneath the parabolic cylinder $z = 1 - x^2$ and within the half-spaces $z \geq 0, y \geq 0$ and $y + z \leq 2$. Compute the flux integral
$$\int\int_S \vec{F} \cdot d\vec{S}$$
using the outward pointing normal vector to the region $R$.

9. Let $A$ and $B$ be non-zero complex $n \times n$ matrices. Prove that if $AB = BA$ then $A$ and $B$ share a common eigenvector.

10. Recall that a function $f : [a, b] \to \mathbb{R}$ is called *Lipschitz* if there exists a constant $C \geq 0$ such that
$$|f(x) - f(y)| \leq C|x - y|$$
for all $x, y \in [a, b]$. Suppose $f$ and $g$ are Lipschitz functions defined on $[a, b]$.
    (a) Prove that $f + g$ is Lipschitz.
    (b) Prove that $fg$ is Lipschitz.

11. Let $X$ be a metric space with distance function $d$. Prove that every Cauchy sequence in $X$ is bounded.

12. Let $\alpha(t) = (t \cos t, t \sin t, t)$ describe a space curve.
    (a) Find an expression, as a function of $t$, for the cosine of the angle between $\alpha'(t)$ and $\alpha''(t)$.
    (b) Prove that that $\alpha'(t)$ is never parallel to $\alpha''(t)$, and that there exists a unique value of $t$ for which $\alpha'(t)$ is orthogonal to $\alpha''(t)$.