Signature | Printed Name

Preliminary Examination, Part I

Thursday, April 28, 2016 | 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete. Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight.

Score	
1	
2	
3	
4	
5	
6	
Total 1–6	
Total 7–12	
Total	

1. Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the closed interval $-1/2 \leq x \leq 1/2$. State any results you are using.

2. Find an orthogonal matrix R that diagonalizes the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3. Let $f(x)$ be a C^∞ real-valued function on $\mathbb{R}$ satisfying $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

   a) Show that at any point x the graph of $y = f(x)$ lies above its tangent line.

   b) If f is bounded above and below, show that $f(x) =$ constant.

4. Let n be a positive integer.

a) Prove that every non-zero element of the ring $\mathbb{Z}/n\mathbb{Z}$ is either a unit or a zero-divisor.

b) For which values of n does $\mathbb{Z}/n\mathbb{Z}$ have the property that every non-zero element is either a unit or is nilpotent (i.e. some power of the element equals zero)?

5. Let $P_1, \ldots, P_k$ be distinct points in $\mathbb{R}^2$.

a) Prove that there is a unique point X_0 in $\mathbb{R}^2$ at which the function

$$Q(X) = \|X - P_1\|^2 + \cdots + \|X - P_k\|^2$$

on $\mathbb{R}^2$ achieves its minimum value.

b) Is there a point at which this function achieves its maximum value?

6. Let X be a metric space and let $\{x_n\}$ be a convergent sequence of points in X with limit L. Show that the set $\{x_n \mid n \in \mathbb{N}\}$ is compact if and only if some x_n is equal to L.

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Preliminary Examination, Part II

Thursday, April 28, 2016 1:30-4:00

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All problems have equal weight.

Score	
7	
8	
9	
10	
11	
12	
Total 7–12	

7. Evaluate the counterclockwise contour integral $J := \oint_\Gamma x^2y^2\, ds$ along the unit circle Γ centered at the origin. [The parameter ds is arc length].

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that there exist $v, w \in \mathbb{R}^2$ such that $T(v) = v$ and $T(w) \neq w$. Show that T is diagonalizable if and only if it has an eigenvalue unequal to 1.

9. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative satisfies the inequality $|g'(x)| \leq M$ for all x in $\mathbb{R}$.

   Show that if $\varepsilon > 0$ is small enough, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \varepsilon g(x)$ is one-to-one and onto.

10. Let G be a group of order 155.

a) Show that G must have a non-trivial proper normal subgroup.

b) Suppose that G (still of order 155) is abelian. Either prove that G is cyclic or give a counterexample.

11. Let Ω be a connected open set in the plane $\mathbb{R}^2$ and let $f(x, y)$ be a C^∞ real-valued function with the property that $\operatorname{grad}(f) = 0$ at every point of Ω. Prove that f is a constant.

12. Let V_0, V_1, V_2 be subspaces of a real vector space V, with V_0 a proper subspace of V_1 and of V_2. Let $S : V_1 \to V_0$ and $T : V_0 \to V_2$ be linear transformations.

a) If V is finite dimensional, show that $T \circ S : V_1 \to V_2$ is neither injective nor surjective.

b) Does the same conclusion necessarily hold if V is infinite dimensional? Give either a proof or counterexample.