Preliminary Examination Solutions

Thursday, April 28, 2016 9:30-12:00

1. Let $\{a_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on the closed interval $-1/2 \le x \le 1/2$. State any results you are using.

   Solution: Since $a_n \to 0$, the sequence is bounded, so for some M we have $|a_n| \le M$ for all $n = 0, 1, 2, \ldots$. Thus if $|x| \le 1/2$ we know that $|a_n x^n| \le M/2^n$. Since the geometric series $\sum M/2^n$ converges, by the comparison test this series converges absolutely. Moreover, by the Weierstrass M-test the series $\sum a_n x^n$ converges uniformly for $|x| \le 1/2$.

   More directly (without using the Weierstrass test), let $c = 1/2$. If $|x| \le c$ then for any integer k

$$\left| \sum_{j=k+1}^{\infty} a_j x^j \right| \le \sum_{j=k+1}^{\infty} |a_j| c^j \le M \sum_{j=k+1}^{\infty} c^j = M \frac{c^{k+1}}{1-c}$$

   which can be made as small as you wish by choosing k large independently of x. Thus the series converges uniformly.

2. Find an orthogonal matrix R that diagonalizes the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution: We solve for R such that $R^{-1}AR$ is diagonal. That is to find orthonormal eigenvectors.

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 1 & 0 \\ 1 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 \lambda$$

When $\lambda = 2$,

$$(A - 2)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Two linearly independent eigenvectors are $(1, -1, 0)$ and $(0, 0, 1)$.

When $\lambda = 0$,

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

An eigenvector is $(1, 1, 0)$. Making the basis orthonormal, we get

$$R = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Let $f(x)$ be a C^∞ real-valued function on $\mathbb{R}$ satisfying $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

a) Show that at any point x the graph of $y = f(x)$ lies above its tangent line.

Solution 1: Let $A = (a, f(a))$ and $B = (b, f(b))$ be two different points in the graph, assume $b > a$. We will prove B is lying above the tangent line at A. The other direction can be proved similarly. To do so, we only need to prove that the slope of AB is greater or equal than the slope of the tangent line at A. Indeed, slope of AB is equal to $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in [a, b]$ by the mean value theorem. Notice that $f'(c) \geq f'(a)$ due to the fact that f' is monotonically increasing (for $f'' \geq 0$).

Solution 2: Pick any point x_0 and any x. Then by Taylor's Theorem with two terms, there is some c between x_0 and x so that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(c)(x - x_0)^2 \\ &\geq f(x_0) + f'(x_0)(x - x_0) \end{aligned}.$$

b) If f is bounded above and below, show that $f(x) =$ constant.

Solution 1: If not, then there is a point a such that $f'(a) \neq 0$. WLOG, assume $f'(a) > 0$. We claim that f is not bounded above, which is a contradiction. Indeed, by the mean value theorem, for each $x > a$ there is constant $c \in [a, x]$ such that $f(x) - f(a) = (x - a)f'(c) \geq (x - a)f'(a) \to \infty$ as $x \to \infty$

Solution 2: By contradiction, say at some point a $f'(a) \neq 0$. Say $f'(a) > 0$. Since the graph lies above its tangent line at $x = a$, in this case $\lim_{x\to+\infty} f(x) = +\infty$.
If $f'(a) < 0$, then $\lim_{x\to-\infty} f(x) = +\infty$.
Either of these contradict the boundedness of f.

4. Let n be a positive integer.

a) Prove that every non-zero element of the ring $\mathbb{Z}/n\mathbb{Z}$ is either a unit or a zero-divisor.

Solution 1: If $a \in \mathbb{Z}$ is relatively prime to n, there exist $x, y \in \mathbb{Z}$ such that $ax + ny = 1$. Thus $ax - 1$ is divisible by n, so the class $\overline{x}$ of x in $\mathbb{Z}/n\mathbb{Z}$ is an inverse of $\overline{a}$ in $\mathbb{Z}/n\mathbb{Z}$.

Otherwise, the gcd d of a, n is > 1, and we may write $a = da'$ and $n = dn'$. Then $\overline{a} \cdot \overline{n'} = \overline{a'n} = \overline{0}$ with $\overline{n'} \neq \overline{0}$ and so $\overline{a}$ is a zero-divisor.

Solution 2: For any nonzero element $a \in \mathbb{Z}/n\mathbb{Z}$, consider the group homomorphism from $(\mathbb{Z}/n\mathbb{Z}, +)$ to $(\mathbb{Z}/n\mathbb{Z}, +)$ given by

$$f_a : x \mapsto ax.$$

If 1 is in the image, then there is $b \in \mathbb{Z}/n\mathbb{Z}$ such that $ab = 1$, so a is a unit. Otherwise 1 is not in the image, in which case $f_a : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is not surjective. Since $\mathbb{Z}/n\mathbb{Z}$ is finite, f_a has a nontrivial kernel. So $ac = 0$ for any $0 \neq c \in \ker f_a$, and a is thus a zero-divisor.

b) For which values of n does $\mathbb{Z}/n\mathbb{Z}$ have the property that every non-zero element is either a unit or is nilpotent (i.e. some power of the element equals zero)?

Solution: This holds iff n is a power of a prime number.

For the forward direction, suppose $n = p^k$ with p a prime and k a positive integer, and let $a \in \mathbb{Z}$ with $\bar{a} \neq 0 \in \mathbb{Z}/n\mathbb{Z}$. If $p \nmid a$, then p^k and a are coprime, so there exist $x, y \in \mathbb{Z}$ with $ax + p^k y = 1$, and $\bar{a}$ is unit in $\mathbb{Z}/n\mathbb{Z}$. If instead $p | a$, then $p^k | a^k$, and so $\bar{a}^k = 0$ in $\mathbb{Z}/n\mathbb{Z}$; i.e. $\bar{a}$ is nilpotent.

Conversely, if n has at least two distinct prime factors, then there are integers $a, b > 1$ with $n = ab$ and $(a, b) = 1$. The nonzero cosets $\bar{a}, \bar{b} \in \mathbb{Z}/n\mathbb{Z}$ satisfy $\bar{a}\bar{b} = 0$, so $\bar{a}$ is not a unit. Since $b \nmid a^k$ for any k, $\bar{a}$ is not nilpotent either. So the property fails.

5. Let $P_1, \dots, P_k$ be distinct points in $\mathbb{R}^2$.

a) Prove that there is a unique point X_0 in $\mathbb{R}^2$ at which the function

$$Q(X) = \|X - P_1\|^2 + \cdots + \|X - P_k\|^2$$

on $\mathbb{R}^2$ achieves its minimum value.

Solution 1: We complete the square

$$\begin{aligned} Q(X) &= kX \cdot X - \sum_{n=1}^{k} 2P_n \cdot X + \sum_{n=1}^{k} P_n \cdot P_n \\ &= k\left\| X - \frac{1}{k}\sum_{n=1}^{k} P_n \right\|^2 + \sum_{n=1}^{k} P_n \cdot P_n - \frac{1}{k}\|\sum_{n=1}^{k} P_n\|^2 \end{aligned}$$

Thus there is a unique point $X_0 = \frac{1}{k}\sum_{n=1}^{k} P_n$ in $\mathbb{R}^2$ at which the function achieves its minimum.

Solution 2: We find the critical points of Q which are where the first derivative (gradient) of Q is zero.

Write $X = (x, y)$ and $P = (p, q)$. For the function

$$f(x, y) = \|X - P\|^2 = (x - p)^2 + (y - q)^2 = x^2 - 2xp + p^2 + y^2 - 2yq + q^2$$

we have

$$\nabla f = (f_x,\ f_y) = (2x - 2p,\ 2y - 2q).$$

Writing $P_j = (p_j,\ q_j)$ this gives

$$\nabla Q = 2\left(\textstyle\sum_1^k (x - p_j),\ \sum_1^k (y - q_j)\right) = 2(kx - \textstyle\sum_1^k p_j,\ ky - \sum_1^k q_j).$$

Thus $\nabla Q = 0$ at only one point: where $x = \frac{1}{k}\sum_1^k p_j$, $y = \frac{1}{k}\sum_1^k q_j$. That is, $X_0 = \frac{1}{k}\sum_1^k P_j$.

We need to show that this point X_0 is the global minimum of Q. Since $\lim_{\|X\|\to\infty} Q(X) = \infty$, there is some radius R so that if $\|X\| > R$ then $Q(X) > Q(0)$. Since at the point X_0 where Q attains its global minimum $Q(X_0) \le Q(0)$, this global minimum of Q must lie inside the disk $\{\|X\| \le R\}$, which is a compact set. At any local minimum point, $\nabla Q = 0$. But we found only one such point X_0 so it must be where Q has its global minimum value.

b) Is there a point at which this function achieves its maximum value?

Solution: No! $Q(X) \to \infty$ as $\|X\| \to \infty$

6. Let X be a metric space and let $\{x_n\}$ be a convergent sequence of points in X with limit L. Show that the set $\{x_n \,|\, n \in \mathbb{N}\}$ is compact if and only if some x_n is equal to L.

Solution: Suppose that $L = x_{n_0}$ for some $n_0 \in \mathbb{N}$, and let $\{U_i\}_{i\in I}$ be an open cover of $x_n \,|\, n \in \mathbb{N}$. Then there is $i_0 \in I$ such that $L \in U_{i_0}$. Since U_{i_0} is open, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in U_{i_0}$. For each $1 \leq m < N$ find U_{i_m} that contains the element x_m. Then $\{U_{i_n} | \, 0 \leq n \leq N\}$ is an open subcover of $\{U_i\}_{i\in I}$.

Conversely suppose that L does not belong to the sequence. Then $U_n = \{x \in X \,|\, d(x, L) > \frac{1}{n}\}$ is an open cover of the set (it is an open cover of $X \setminus \{L\}$), that does not have any finite subcover. For otherwise the sequence would be contained in one of the U_n's and could not converge to L. It follows that the set is not compact.

7. Evaluate the counterclockwise contour integral $J := \oint_\Gamma x^2y^2\,ds$ along the unit circle Γ centered at the origin. [The parameter ds is arc length].

Solution: On the unit circle we use the polar coordinate parameterization $x = \cos\theta$, $y = \sin\theta$. Then

$$x^2y^2 = \cos^2\theta\sin^2\theta \quad \text{and} \quad ds = \sqrt{x'^2 + y'^2}\,d\theta = d\theta.$$

The integral is thus $J = \int_0^{2\pi} \cos^2\theta\sin^2\theta\,d\theta$.

The identity $\sin 2\theta = 2\sin\theta\cos\theta$ followed by the substitution $\phi = 2\theta$ gives

$$J = \int_0^{2\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta = \frac{1}{8}\int_0^{4\pi} \sin^2\phi\,d\phi = \frac{1}{4}\int_0^{2\pi} \sin^2\phi\,d\phi.$$

But

$$\int_0^{2\pi} \sin^2\phi\,d\,\phi = \int_0^{2\pi} \frac{1-\cos 2\phi}{2}\,d\phi = \left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{2\pi} = \pi$$

so

$$J = \frac{\pi}{4}.$$

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that there exist $v, w \in \mathbb{R}^2$ such that $T(v) = v$ and $T(w) \neq w$. Show that T is diagonalizable if and only if it has an eigenvalue unequal to 1.

   Solution: $\Rightarrow$: Suppose for a contradiction that T has all eigenvalues equal to one. Since T is diagonalizable, this implies that T is the identity. But this contradicts the assumption that $T(w) \neq w$.

   $\Leftarrow$: As $T(v) = v$ and v is not zero, one is an eigenvalue of T. So if T also has an eigenvalue not equal to one, T is diagonalizable, since any 2×2 matrix with two distinct eigenvalues is diagonalizable.

9. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative satisfies the inequality $|g'(x)| \leq M$ for all x in $\mathbb{R}$.

   Show that if $\varepsilon > 0$ is small enough, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \varepsilon g(x)$ is one-to-one and onto.

   Solution: We may and shall assume that $M \geq 1$, and choose $\varepsilon \leq \frac{1}{2M}$. It follows that
$$f'(x) = 1 + \varepsilon g'(x) \geq 1 - \varepsilon M \geq \frac{1}{2},$$
   for every $x \in \mathbb{R}$. The Mean Value Theorem then implies that
$$|f(x) - f(y)| = |f'(\xi)||x - y| \geq \frac{1}{2}|x - y|.$$
   This proves injectivity.

   Applying the Mean Value Theorem one more time we get
$$f(x) - f(0) = f'(\xi)x.$$
   This last quantity is $\geq \frac{x}{2}$, for $x \geq 0$, and $\leq \frac{x}{2}$ for $x \leq 0$. In any case it follows that
$$\lim_{x \to \pm\infty} f(x) = \pm\infty,$$
   and the intermediate value theorem proves that f is onto.

10. Let G be a group of order 155.

a) Show that G must have a non-trivial proper normal subgroup.

Solution: The prime factorization of 155 is $5 \cdot 31$. By the Sylow theorem, the number of Sylow 31-groups divides 5 and is congruent to 1 mod 31, so it must be 1. That is, there is a unique subgroup of order 31, so it is normal.

b) Suppose that G (still of order 155) is abelian. Either prove that G is cyclic or give a counterexample.

Solution: By the Fundamental Theorem of Finite Abelian Groups, a finite abelian group is a product of cyclic groups of prime power order. So a group of order $5 \cdot 31$ is isomorphic to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/31\mathbb{Z}$. By the Chinese Remainder Theorem, this group is isomorphic to $\mathbb{Z}/(5 \cdot 31)\mathbb{Z}$ and so is cyclic.

11. Let Ω be a connected open set in the plane $\mathbb{R}^2$ and let $f(x, y)$ be a C^∞ real-valued function with the property that $\text{grad}\,(f) = 0$ at every point of Ω. Prove that f is a constant.

Solution: The Mean Value Theorem implies right away that if $x \in \Omega$, then there is an open ball B_r centered at x, lying entirely in Ω, such that $f(y) = f(x)$ for every $y \in B_r$. This shows that if we fix $x_0 \in \Omega$, then the set $\{y \in \Omega \mid f(y) = f(x_0)\}$ is open. Since it is obviously closed (in Ω, being the preimage of a closed set), and nonempty it has to be all of Ω.

12. Let V_0, V_1, V_2 be subspaces of a real vector space V, with V_0 a proper subspace of V_1 and of V_2. Let $S : V_1 \to V_0$ and $T : V_0 \to V_2$ be linear transformations.

   a) If V is finite dimensional, show that $T \circ S : V_1 \to V_2$ is neither injective nor surjective.

      Solution1: The rank of $T \circ S$ is less than or equal to the rank of T, which is less than or equal to the dimension of V_0, which is less than the dimension of V_1 and is also less than the dimension of V_2. Therefore $T \circ S$ is neither injective nor surjective.

      Solution 2: If V is finite dimensional, also V_0, V_1, V_2 are finite dimensional. As V_0 is properly contained in V_1 and V_2 we have $\dim(V_0) < \dim(V_1)$ and $\dim(V_0) < \dim(V_2)$. Therefore $S\colon V_1 \to V_0$ cannot be injective and $T\colon V_0 \to V_2$ cannot be surjective. Thus the composition $T \circ S$ can neither be injective nor surjective.

   b) Does the same conclusion necessarily hold if V is infinite dimensional? Give either a proof or counterexample.

      Solution 1: Counterexample: Let $V_1 = V_2 = V$ be a vector space spanned by a countable basis $e_0, e_1, e_2, \ldots$; V_0 the proper subspace spanned by $e_1, e_2, \ldots$; $S(e_i) := e_{i+1}, i = 0, 1, \ldots$ (right shift); and $T(e_i) := e_{i-1}, i = 1, 2, \ldots$ (left shift). Then S, T are isomorphisms, as is $T \circ S$.

      Solution 2: Consider the following counterexample:
      Let $V = \mathbb{R}^{\mathbb{N}} = \{(a_n)_{n\in\mathbb{N}}\}$ be the vector space of all sequences (with componentwise addition and scalar multiplication). Let $V_1 = V$ and

$$V_2 = \{(a_n)_{n\in\mathbb{N}} |\ a_0 = 0\},$$

$$V_0 = \{(a_n)_{n\in\mathbb{N}} |\ a_0 = 0, a_1 = 0\}.$$

      Define $S\colon V_1 \to V_0$ by $S((a_0, a_1, \ldots)) = (0, 0, a_0, a_1, \ldots)$ and $T\colon V_0 \to V_2$ by $T((0, 0, a_2, a_3, \ldots)) = (0, a_2, a_3, \ldots)$. Then S and T are bijective and therefore also $T \circ S$ is bijective.