Spring prelim answers

1.List all the finite fields (up to isomorphism) of order less than or equal to 10. Show that the ones you list exist, and no others.

Solution: There is a unique finite field of each prime power order, so: 2,3,4,5,7,8,9. We show: (1) the order must be a prime power $q = p^k$; (2) a field of each prime power cardinality q exists; and (3) it is unique.

(1) Let F be a finite field and $\mathbf{1} \in F$ the unit element. Since F is finite there is a positive integer c such that $c\mathbf{1} = 0 \in F$. If $c = ab$ then $0 = c\mathbf{1} = a\mathbf{1}b\mathbf{1}$ so $a\mathbf{1} = 0$ or $b\mathbf{1} = 0$. So the smallest such c must be a prime p, and F is then a vector space over the prime field F_p, hence its cardinality is a power of p.

(2) Conversely, let $\overline{F}$ be an algebraic closure of F_p. If $q = p^k$, the set

$$\{x \in \overline{F} | x^q = x\}$$

is closed under addition (since $x^q + y^q = (x + y)^q$), multiplication and inversion, so it is a subfield F_q. Since the polynomial $x^q - x$ has no repeated roots, the cardinality of F_q is q.

(3) Since any filed of cardinality q is the set of elements in its algebraic closure satisfying $x^q = x$, the uniqueness of the algebraic closure implies uniqueness of F_q.

2. (a) In the polynomial ring $\mathbf{Z}[x]$, is the ideal generated by $x^4 - 1$ and $2x^3 - 2x$ principal?

(b) Same question in the polynomial ring $\mathbf{Q}[x]$.

(c) Same question in the polynomial rings $\mathbf{Z}[x, y]$ and $\mathbf{Q}[x, y]$.

Solution: (a) No: such a generator must be of the form $(x^2 - 1)f(x)$, and then $f(x)$ must generate the ideal $I := (x^2 + 1, 2x)$, in particular $f(x)$ must divide $x^2 + 1$ and $2x$, so it must be invertible, but I is non trivial, since it is contained in the kernel of the surjective homomorphism $\mathbf{Z}[x] \to \mathbf{Z}/(2)$ sending x to 1. (b) Yes, in $\mathbf{Q}[x]$ we can take the above $f(x)$ to equal 1, since $1 = 1(x^2 + 1) - (\frac{x}{2})2x) \in (x^2 + 1, 2x)$. (c) No change. (The analogue of (b) is trivial, and for (a) send y to 0.)

3. Use the ε-δ definition to prove that the first derivative of $f(x) = x^3$ is $f'(x) = 3x^2$.

Solution: We must show that for any given x_0 and for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $\left|\frac{x^3-x_0^3}{x-x_0} - 3x_0^2\right| < \varepsilon$. An easy computation shows that:

$$\left|\frac{x^3 - x_0^3}{x - x_0} - 3x_0^2\right| = \left|(x^2 + x_0x + x_0^2) - 3x_0^2\right| = \left|x^2 - x_0^2 + x_0x - x_0^2\right| \leq$$
$$\leq \left|x^2 - x_0^2\right| + |x_0||x - x_0| \leq |x - x_0||x + x_0| + |x_0||x - x_0| \leq$$
$$\leq |x - x_0|(|x - x_0| + 2|x_0|) + |x_0||x - x_0|, \quad (1)$$

where the last inequality comes from $|x + x_0| \leq |x| + |x_0| \leq (|x - x_0| + |x_0|) + |x_0| = |x - x_0| + 2|x_0|$, since $||x| - |x_0|| \leq |x - x_0|$ by the triangle inequality. Thus, choosing any

$$0 < \delta < \frac{-3|x_0| + \sqrt{9|x_0|^2 + 4\varepsilon}}{2},$$

it follows that $\delta(\delta + 2|x_0|) + |x_0|\delta = \delta^2 + 3|x_0|\delta < \varepsilon$ and hence by (1), the desired inequality $\left|\frac{x^3-x_0^3}{x-x_0} - 3x_0^2\right| < \varepsilon$ holds whenever $0 < |x - x_0| < \delta$.

4. Prove that for all $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that all $n \times n$ matrices A with $\|A - \mathrm{Id}\| < \varepsilon_k$ have a k^{th} *root*, that is, an $n \times n$ matrix $\sqrt[k]{A}$ such that $(\sqrt[k]{A})^k = A$.

Solution: Identify the vector space of $n \times n$ matrices with $\mathbb{R}^{n^2}$, and consider the map $f(A) = A^k$. The coordinates of this map are smooth functions (polynomial) and hence f is smooth. The linearization of f at the identity matrix Id is given by

$$\mathrm{d}f(\mathrm{Id})X = kX, \qquad X \in \mathbb{R}^{n^2}.$$

In particular, $\mathrm{d}f(\mathrm{Id})$ is invertible, hence by the Inverse Function Theorem, f is locally invertible near $\mathrm{Id} \in \mathbb{R}^{n^2}$. Thus, there exists $\varepsilon_k > 0$ such that if $\|A - \mathrm{Id}\| < \varepsilon_k$, then $\sqrt[k]{A} := f^{-1}(A)$ is a k^{th} root of A.

5. . Let N be a positive integer. Prove that

$$\frac{1}{2} + \log N < \sum_{k=1}^{N} \frac{1}{k} \leq 1 + \log N.$$

Solution: This is a standard problem which appears in many calculus books. A partial sum of the harmonic series can be viewed geometrically as the sum of the areas of boxes of width one and height $1, 1/2, 1/3, \ldots, 1/N$, which can be approximated by the area under the curve $y = \ln x$ from 1 to N. The difference is the sum of the areas of a bunch of slivers of width 1 above the graph of $y = 1/x$, together with the last box of height $1/N$. If you slide all these slivers left, they all fit in a box of width 1 and height 1, and the concavity of the graph shows the sum of the areas of the slivers is larger than $1/2$. See for example the picture on p. 595 of Thomas' Calculus, applied to the case $f(x) = \ln x$.

6. For n a positive integer, let $\phi(n)$ denote the number of integers k, $1 \leq k < n$, which are relatively prime to n. Prove that

$$\phi(n) = n \prod_{p|n} (1 - \frac{1}{p}),$$

where the product is over all distinct primes p dividing n.

Solution: We use a standard inclusion-exclusion argument. The number of integers relatively prime to n and between 1 and n is the number of integers between 1 and n (there are n of these), minus all multiples of p for each p dividing n (there are n/p of these), plus all multiples of pq for all pairs of distinct primes p, q dividing n (there are $n/(pq)$ of these), etc., or

$$n - \sum_{p|n} n/p + \sum_{\substack{p<q \\ p,q|n}} n/(pq) - \ldots = n \prod_{p|n} (1 - 1/p).$$

7. Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Find an orthonormal basis of the column space of A.

Solution: Let a_j denote the j-th column vector of A. Let $v_1 = a_1$ and

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{\|v_1\|} v_1 = (2, 0, -1, 1)^T.$$

Let

$$v_3 = a_3 - \frac{a_3 \cdot v_1}{\|v_1\|} v_1 - \frac{a_3 \cdot v_2}{\|v_2\|} v_2 = (-1, 1, 0, 2)^T.$$

Then an orthonormal basis is

$$\{\frac{1}{\sqrt{6}}(1, 1, 2, 0)^T, \frac{1}{\sqrt{6}}(2, 0, -1, 1)^T, \frac{1}{\sqrt{6}}(-1, 1, 0, 2)^T\}$$

8. Let A be a $n \times n$ matrix. Let $\{S_1, \ldots S_k\}$ be a collection of eigenvectors of A with $\lambda_1, \ldots \lambda_k$ as the corresponding eigenvalues. Prove that if $\lambda_i \neq \lambda_j$ for all $1 \leq i < j \leq k$, then $\{S_1, \ldots, S_k\}$ is linearly independent.

Solution: Suppose there are constants $\{c_j\}_{1 \leq j \leq k}$ such that

$$\sum_{j=1}^{k} c_j S_j = 0.$$

For $1 \leq l \leq k$, define

$$B_l = \prod_{1 \leq j \leq k, j \neq l} (A - \lambda_j I).$$

Then

$$0 = B_l \sum_{j=1}^{k} c_j S_j = c_l \prod_{1 \leq j \leq k, j \neq l} (\lambda_l - \lambda_j) S_l$$

Since $\lambda_i \neq \lambda_j$ for all $i \neq j$, $c_l = 0$ for all $1 \leq l \leq k$.

9. Let (X,d) be a compact metric space. Suppose that $f : X \to X$ satisfies $d(f(x), f(y) < d(x, y)$ for $x \neq y$. Show that for any $x \in X$, the sequence defined by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to a unique fixed point of f.

Solution: If there were two fixed points x and y, then $d(x, y) < d(x, y)$, so there is at most one. The function $d(f(x), x)$ is continuous and so achieves a minimum at some point y. If the mimimum is not zero, then $f(y) \neq y$,, in which case $d(f(y), f^2(y))$ is smaller, a contradiction. Therefore it is zero, i.e. $f(y) = y$.

The same thing applies to the closure A of the set of points $\{x_1, ..., x_n, ...\}$, as this closure is also compact and invariant under f. So, from above, the restriction of f to A will have a fixed point. Hence $y \in A$. THerefore, given $\epsilon > 0$, there exists N such that $d(x_N, y) < \epsilon$. For $m > N$,

$$d(x_m, y) < d(x_{m-1}, y) < ... < d(x_{N+1}, y) < d(x_N, y) < \epsilon .$$

Thus the sequence converges to y.

10. A topological property is one that is invariant under homeomorphism, i.e. if two spaces are homeomorphic and one has the property, so does the other. Explain with a proof or counterexample which of the following properties of a metric space are or are not topological invariants: a. Compactness, b. Connectedness, c. Boundedness d. Completeness.

Solution. a. If $f : X \to Y$ is continuous and X is compact, so is $f(Y)$. Namely, for any collection $\mathfrak{A}$ of open sets covering $f(X)$, the sets $f^{-1}(U)$, $U \in \mathfrak{A}$ cover X. Therefore, a finite set of these, $f^{-1}U_1, ..., f^{-1}U_n$ also cover X; therefore $f(X) \subset U_1 \cup ... \cup U_n$. So as each of two homeomorphic spaces are the image of the other under a continuous map, if one is compact so is the other.

b. Similarly, it suffices to show that if $f : X \to Y$ is continuous and X is connected, so is $f(X)$. If $f(X)$ is the disjoint union of two non-empty open sets, then the inverse images will be two non-empty disjoint open sets whose union os all of X.

c. Not a topological invariant. For example, $(0,1)$ and $\mathbb{R}$ with the usual metric are homeomorphic. Alternatively, if (X,d) is a metric space, and $d'(x,y) = \frac{d(x,y}{1+d(x,y)}$, then (X,d) and $(X.d')$ have the same topology, i.e. the identity map is a homeomorphism, and $d'(x,y) \leq 1$.

d. $\mathbb{R}$ is complete but $(0,1)$ is not.

11. Suppose $f : [-1,1] \to \mathbb{R}$ is continuous on the closed interval $[-1,1]$, and twice differentiable on the open interval $(-1,1)$. Suppose also that $f(-1) = 7, f(0) = 1$ and $f(1) = 1$. Prove that there exists $c \in (-1,1)$ such that $f^{(2)}(c) = 6$.

Solution: The (unique) polynomial of degree two that passes through those points is

$$p(x) = 3x^2 - 3x + 1.$$

It follows that the function $f - p$ vanishes at $-1, 0$ and 1. Applying Rolle's Theorem twice, it follows that there exists a point $c \in (-1,1)$ such that the second derivative of $f - p$ vanishes at c. Since the second derivative of p is constant equal to 6, it follows that $f^{(2)}(c) = 6$.

12. Compute the following limit if it exists and justify your conclusion:

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n (1-x^5)^{\frac{1}{5}}\, dx.$$

Solution: Remark that the function $(1-x^5)^{\frac{1}{5}}$ is continuous on $[0,1]$ and equal to zero at $x = 1$. It follows that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|(1-x^5)^{\frac{1}{5}}| = (1-x^5)^{\frac{1}{5}} < \varepsilon/2$ for $x \in [1-\delta, 1]$. On the other hand if $x \in [0, 1-\delta]$, then $0 \leq (n+1)x^n \leq (n+1)(1-\delta)^n \to 0$ as $n \to \infty$. It follows that there exists $N \in \mathbb{N}$ such that $(n+1)(1-\delta)^n < \varepsilon/2$, for every $n \geq N$. It follows that

$$0 \leq \int_0^1 (n+1)x^n(1-x^5)^{\frac{1}{5}}\, dx = \int_0^{1-\delta} (n+1)x^n(1-x^5)^{\frac{1}{5}}\, dx + \int_{1-\delta}^1 (n+1)x^n(1-x^5)^{\frac{1}{5}}\, dx \leq$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\int_{1-\delta}^1 (n+1)x^n\, dx \leq \varepsilon,$$

for every $n \geq N$. So we proved that the limit is zero.