Problem 1. Let \( \{a_n\} \) be a sequence of real numbers such that \( \lim_{n \to \infty} a_n = 0 \). Prove that the series
\[
\sum_{n=0}^{\infty} a_n x^n
\]
converges uniformly on the closed interval \([-\frac{1}{2}, \frac{1}{2}]\). State any results you are using.

Solution. Since the sequence \( \{a_n\} \) converges, it is bounded. Indeed, since \( a_n \to 0 \), there is a positive number \( N \), such that \( |a_n| < 1 \) for \( n > N \). For \( M = \max\{|a_1|, \ldots, |a_N|, 1\} \) we have that \( |a_n| \leq M \) for all \( n \in \mathbb{N} \). This means that for \( x \in [-\frac{1}{2}, \frac{1}{2}] \)
\[
|a_n x^n| = |a_n| \cdot |x|^n \leq \frac{M}{2^n}.
\]
Observe that the series of the real numbers on the right hand side is geometric and hence it converges:
\[
\sum_{n=0}^{\infty} a_n x^n \leq \sum_{n=0}^{\infty} |a_n x^n| \leq \sum_{n=0}^{\infty} \frac{M}{2^n} = M,
\]
Hence, by the Weierstrass M-test, the series \( \sum_{n=0}^{\infty} a_n x^n \) converges absolutely and uniformly on \([-\frac{1}{2}, \frac{1}{2}]\). \( \square \)

Problem 2. Find an orthogonal matrix \( R \) that diagonalizes the matrix
\[
A = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

Solution. In order to construct the matrix \( R \), we have to compute the eigenvalues of the matrix \( A \), and then find an orthonormal basis of \( \mathbb{R}^3 \) consisting of eigenvectors for the respective eigenvalues of \( A \), i.e.

one needs to:

- compute the characteristic polynomial \( \chi_A(x) \) of \( A \), and its roots,
- solve the linear system \( Ax = \lambda x \), for each eigenvalue \( \lambda \), and find bases for the respective eigenspaces,
- perform Gram-Schmidt process to replace the basis with an orthonormal one.

All of the above steps are fairly standard, so we will try to avoid them, and try to get the solutions to those questions almost by just looking at our matrix.

First observe that the matrix is already separated in a \( 2 \times 2 \) and a \( 1 \times 1 \) block. So if we think of \( \mathbb{R}^3 \) as \( \mathbb{R}^2 \oplus \mathbb{R} \), with the first component being the \( xy \)-plane, and the second component being the \( z \)-axis, the action of \( A \) on \( \mathbb{R}^3 \) splits into the action of \( A_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \) on the \( xy \)-plane, and a multiplication by \( 2 \) on the \( z \)-axis.

- This immediately implies that \( 2 \) is an eigenvalue for \( A \), with \( u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) a respective eigenvector.

\( ^{0} \)This text contains comments, details, or even alternative paths to solve some of the problems. A lot of the times, the detail in which a solution is presented, is to demonstrate facts that should be standard knowledge, and not to indicate how detailed an examinee’s solution has to be. Be aware, (hopefully minor) typos might exist!
• Since $u_1$ is already normal, and the $z$-axis is already orthogonal to the $xy$-plane, we just need to find an orthogonal matrix diagonalizing $A_1$.

Now observe that $A_1$ has an obvious vector in the kernel (i.e. a 0-eigenvector): $A_1(\frac{1}{1}) = 0$. So, if $A$ is diagonalizable, the orthogonal direction to $\frac{1}{1}$ in $\mathbb{R}^2$, i.e. span$(\frac{1}{1})$, has to be an eigenspace with respect to some eigenvalue. We compute:

$$
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-2
\end{pmatrix}
= 2 \cdot 
\begin{pmatrix}
1 \\
-1
\end{pmatrix},
$$

i.e. $\frac{1}{1}$ is a 2-eigenvector. After normalizing, we get an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors:

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

So the desired matrix $R$ is

$$
R = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad R^{-1} = R' = R.
$$

and the desired diagonalization is obtained:

$$
R^{-1}AR = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
$$

\[\square\]

**Problem 3.** Let $f(x)$ be a $C^\infty$-real-valued function on $\mathbb{R}$, satisfying $f''(x) \geq 0$ for all $x \in \mathbb{R}$.

(a) Show that at any point $x$, the graph of $y = f(x)$ lies above the tangent line.

(b) If $f$ is bounded above and below, show that $f(x)$ is constant.

**Solution.** (a) Since $f(x) \geq 0$, we have that $f'$ is monotone increasing. Indeed, by the mean value theorem for $f'$, if $x < y$, there is a $\xi \in (x, y)$, such that:

$$
\frac{f'(y) - f'(x)}{y - x} = f''(\xi) \geq 0
$$

Since $y - x > 0$, we have that $f'(y) - f'(x) \geq 0$, i.e. $f(y) \geq f(x)$. Now fix a point $p \in \mathbb{R}$, and consider the tangent line at $(p, f(p))$, which it given by

$$
t_p : \mathbb{R} \to \mathbb{R}, \quad t_p(x) = f(p) + f'(p)(x - p).
$$

We consider the function $g(x) = f(x) - t_p(x)$, and wish to prove that $g(x) \geq 0$ for any $x \in \mathbb{R}$. We check the cases:

- $x = p$: Then $t_p(x) = f(x)$, i.e. $g(x) = 0$.
- $x > p$: Then the mean value theorem for $g$ gives that there is a $\xi_1 \in (p, x)$ such that:

$$
\frac{g(x)}{x - p} = \frac{g(x) - g(p)}{x - p} = g'(\xi_1) = f'(\xi_1) - t_p'(\xi_1) = f'(\xi_1) - f'(p) \geq 0,
$$

and hence $g(x) \geq 0$.
- $x < p$: Similarly, by the mean value theorem, there is a $\xi_2 \in (x, p)$ such that:

$$
\frac{g(x)}{x - p} = \frac{g(x) - g(p)}{x - p} = g'(\xi_2) = f'(\xi_2) - t_p'(\xi_2) = f'(\xi_2) - f'(p) \leq 0,
$$

and hence $g(x) \geq 0$. 

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(b) Seeking contradiction, assume that \( f \) is non-constant. As a result, there is a point \( p \in \mathbb{R} \), such that \( f'(p) \neq 0 \). From (a) we know that \( f \geq t_p \).

- If \( f'(p) > 0 \), we have that
  \[
  \lim_{x \to +\infty} f(x) \geq \lim_{x \to +\infty} t_p(x) = +\infty,
  \]
  which contradicts that \( f \) is bounded above.

- If \( f'(p) < 0 \), we have that
  \[
  \lim_{x \to -\infty} f(x) \geq \lim_{x \to -\infty} t_p(x) = +\infty,
  \]
  which again contradicts that \( f \) is bounded above.

As a result \( f \) is constant. We did not need the fact that \( f \) is bounded below.

\[\square\]

**Problem 4.** Let \( n \) be a positive integer.

(a) Prove that every non-zero element of the ring \( \mathbb{Z}/n\mathbb{Z} \) is either a unit or a zero-divisor.

(b) For which values of \( n \) does \( \mathbb{Z}/n\mathbb{Z} \) have the property that every non-zero element is either a unit or is nilpotent (i.e. some power of the element equals zero)?

**Solution.** (a) Let \( [k]_n \in \mathbb{Z}/n\mathbb{Z} \) and choose a representative \( k \in \mathbb{Z} \). We consider the integer \( d = \text{gcd}(k, n) \) and examine the following cases:

- \( d = 1 \): By Euclid’s algorithm, there exist integers \( a, b \in \mathbb{Z} \) such that \( ak + bn = d = 1 \). Hence
  \[
  [a]_n[k]_n = [ak]_n = [1 - bn]_n = [1]_n - [bn]_n = [1]_n,
  \]
  i.e. \([k]_n\) is invertible.

- \( d \neq 1 \): Write \( n = n' \cdot d \) and \( k = k' \cdot d \), and observe that \( n \) does not devide \( n' \), i.e. \([n']_n \neq [0]_n\).

  Now we compute:
  \[
  [n']_n[k]_n = [n'k'd]_n = [n'd]_n[k']_n = [n]_n[k']_n = [0]_n,
  \]
  i.e. \([k]_n\) is a zero-divisor.

(b) Let’s actually try to figure out the correct condition. Then every prime number \( p \), which is not a unit. If we assume that every non-unit is nilpotent, i.e. there is a large natural number \( N \), such that \([p^N]_n = [0]_n\), i.e. \( n \) divides \( p^N \), which means that \( n \) has to be a power of that prime \( p \).

Conversely, assume that \( n \) is a power of prime, i.e. \( n = p^r \) for some prime \( p \) and some \( r \in \mathbb{N} \). Then for every element \([k]_n\), which is not a unit, we have that \( \text{gcd}(k, n) = p^s > 1 \), for some \( 0 < s \leq r \), i.e. \( p^s \) divides \( k \). Then, observe that \( p^{Ns} \) divides \( k^N \), but for \( N \) large enough, \( r \leq Ns \), and as a result \( n = p^r \) divides \( k^N \), i.e. \([k^N]_n = 0 \iff [k]_n \) is nilpotent.

\[\square\]

**Problem 5.** Let \( P_1, \ldots, P_k \) be distinct points in \( \mathbb{R}^2 \).

(a) Prove that there is a unique point \( X_0 \) in \( \mathbb{R}^2 \) at which the function

\[
Q(X) = \|X - P_1\|^2 + \cdots + \|X - P_k\|^2
\]

on \( \mathbb{R}^2 \) achieves a minimum value.

(b) Is there a point at which this function achieves its maximum value?

**Solution.** Let’s denote the coordinates of our points \( P_i = (x_i, y_i), 1 \leq i \leq k, X = (x, y) \).

(a) If \( Q_i(X) = \|X - P_i\|^2 \) on \( \mathbb{R}^2 \), then \( Q_i(x, y) = (x - x_i)^2 + (y - y_i)^2 \). As a result

\[
\nabla Q_i(x, y) = (2(x - x_i), 2(y - y_i)).
\]
Problem 6. Let $Q = Q_1 + Q_2 + \cdots + Q_k$, by the linearity of the gradient,
\[
\nabla Q(x, y) = \sum_{i=1}^{k} \nabla Q_i(x, y)
\]
\[
= \sum_{i=1}^{k} (2(x - x_i), 2(y - y_i))
\]
\[
= 2 \left( kx - \sum_{i=1}^{k} x_i, ky - \sum_{i=1}^{k} y_i \right)
\]
\[
= 2 \left( x - \frac{1}{k} \sum_{i=1}^{k} x_i, y - \frac{1}{k} \sum_{i=1}^{k} y_i \right).
\]
So, in order to detect critical points, we solve:
\[
\nabla Q(x, y) = 0 \iff x = \frac{1}{k} \sum_{i=1}^{k} x_i \text{ and } y = \frac{1}{k} \sum_{i=1}^{k} y_i.
\]
In order to identify the nature of the critical point, we can use the second derivative test, so we will compute the Hessian of $Q$. Evidently
\[
\frac{\partial^2 Q}{\partial x \partial y} = \frac{\partial^2 Q}{\partial y \partial x} = 0
\]
\[
\frac{\partial^2 Q}{\partial x^2} = \frac{\partial^2 Q}{\partial y^2} = \frac{2}{k} > 0
\]
Since both eigenvalues of the (diagonal) Hessian are positive, the (unique) critical point is a minimum.

Using the convexity of $Q$ we can verify that the local minimum is a global minimum. Let’s verify that $Q$ is convex. Since, the sum of convex functions is convex, we just have to verify that each $Q_i$ is convex. Let $X, Y \subset \mathbb{R}^2$ and $\lambda \in [0, 1]$. Then:
\[
Q_i(\lambda X + (1 - \lambda)Y) = \|\lambda X + (1 - \lambda)Y - \lambda P_i - (1 - \lambda)P_i\|^2
\]
\[
\leq \|\lambda(X - P_i)\|^2 + \|(1 - \lambda)(Y - P_i)\|^2
\]
\[
\leq \lambda^2 Q_i(X) + (1 - \lambda)^2 Q_i(Y)
\]
\[
\leq \lambda Q_i(X) + (1 - \lambda)Q_i(Y)
\]
Now let $P_0$ be the local minimum we found above, and, seeking contradiction, assume there is a point $X$, such that $Q(X) < Q(P_0)$. Then for $\lambda \in (0, 1),
\[
Q(\lambda X + (1 - \lambda)P_0) \leq \lambda Q(X) + (1 - \lambda)Q(P_0) < \lambda Q(P_0) + (1 - \lambda)Q(P_0) = Q(P_0),
\]
but for $\lambda$ small enough, $\lambda X + (1 - \lambda)P_0$ is in the neighborhood where $P_0$ is a local minimum, and hence $Q(P_0) < Q(\lambda X + (1 - \lambda)P_0)$, which is a contradiction.

(b) Observe that $Q(X) \to \infty$ as $\|X\| \to \infty$, and hence $Q$ does not have a maximum value. Moreover, we know that if there was a point in the open set $\mathbb{R}^2$ where $Q$ achieves a maximum value, since $Q$ is smooth, that point would have to be critical for $Q$. But the computation in (a) shows that the only critical point is a minimum (and $Q$ is not constant). \qed

Problem 6. Let $X$ be a metric space and let $\{x_n\}$ be a convergent sequence of point in $X$ with limit $L$. Show that the set $\{x_n : n \in \mathbb{N}\}$ is compact if and only if some $x_n$ is equal to $L$.

Solution. Clearly, $L$ is a limit point of the set of terms of the sequence.

(⇒) If the set $F = \{x_n : n \in \mathbb{N}\}$ is compact, since $X$ is a metric space (in particular Hausdorff) $F$ has to be closed, and hence contain all of its limit points. In particular $L \in F$, i.e. $L$ is one of the terms of the sequence.

(⇐) Suppose that there is a term $x_{n_0}$ such that $x_{n_0} = L = \lim_{n \to \infty} x_n$. We will prove that the subspace $F = \{x_n : n \in \mathbb{N}\}$ is compact. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an arbitrary open cover of $F$. There is an open set of the cover $U_{i_0}$ such that the subspace $x_{n_0} \in U_{i_0}$. Since $U_{i_0}$ is an open set, there is a $\delta > 0$ such that
\[
B_F(x_{n_0}, \delta) = \{x \in F : d_F(x, x_{n_0}) < \delta\} \subset U_{i_0},
\]
where $d_F$ is the relative metric on $F$ (i.e. the restriction of metric of $X$ on $F$). Since $x_n \to L = x_{n_0}$, there is a $N \in \mathbb{N}$, such that $d_F(x_n, x_{n_0}) < \delta$ for any $n > N$, i.e. $x_n \in U_i$ for any $n > N$. Finally, since $U$ covers $F$, for each of the terms $x_1, \ldots, x_N$, we can find open sets of the cover $U_1, \ldots U_N$, such that $x_i \in U_i$. Now observe that $U_0 = \{U_1, \ldots, U_N, U_{i_0}\} \subset U$ is a finite subcover. As a result, $F$ is compact.

**Problem 7.** Evaluate the counterclockwise contour integral

$$J := \oint_\Gamma x^2 y^2 \, ds$$

along the unit circle $\Gamma$ centered at the origin. [The parameter $ds$ is arc length].

**Solution.** We parametrize $\Gamma$ by $r(t) = (x(t), y(t)) = (\cos(t), \sin(t)) \ t \in [0, 2\pi]$. This is the arc length parametrization, since

$$|r'(t)| = |(\cos'(t), \sin'(t))| = |(-\sin(t), \cos(t))| = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$ 

Now we compute the integral:

$$J := \int_\Gamma x^2 y^2 \, ds = \int_0^{2\pi} x^2(t) y^2(t) |r'(t)| \, dt$$

$$= \int_0^{2\pi} \cos^2(t) \sin^2(t) \, dt$$

$$= \int_0^{2\pi} (\cos(t) \sin(t))^2 \, dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} \sin(2t)\right)^2 \, dt$$

$$= \frac{1}{4} \int_0^{2\pi} \sin^2(2t) \, dt$$

$$= \frac{1}{4} \int_0^{2\pi} \frac{1}{2} (1 - \cos(4t)) \, dt$$

$$= \left[ \frac{t}{8} \sin(4t) \right]_0^{2\pi}$$

$$= \frac{\pi}{4}.$$ 

**Problem 8.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that there exist $v, w \in \mathbb{R}^2$ such that $T(v) = v$ and $T(w) \neq w$. Show that $T$ is diagonalizable if and only if it has an eigenvalue unequal to 1.

**Solution.** For the conclusion to be correct, we need to assume that $v \neq 0$ (of course $w \neq 0$, by the linearity of $T$), otherwise $T(x, y) = (y, 0)$ is a counterexample.

$(\Rightarrow)$ Assume that $T$ is diagonalizable. Since $T$ has an eigenvector $v$ for 1, we know that one of the eigenvalues has to be 1. If the two eigenvalues are identical, this means that there is a basis of $\mathbb{R}^2$ with respect to which the linear transformation $T$ is represented by the matrix $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = \text{id}_{\mathbb{R}^2}$, i.e. $\text{id}_{\mathbb{R}^2} = T$. But this is not possible, since $T(w) \neq w$. Hence $T$ has distinct eigenvalues.

$(\Leftarrow)$ Assume $T$ has an eigenvalue $\lambda \neq 1$, and let $u \in \mathbb{R}^2$ be a corresponding (non-zero) eigenvector. Then the ordered set $(v, u)$ is an ordered basis of $\mathbb{R}^2$. Indeed, since $\dim_{\mathbb{R}} \mathbb{R}^2 = 2$, the set is a basis iff it is linearly independent. This is easy to verify, e.g. otherwise a non-zero multiple $cu$ of $u$ is in the eigenspace with respect to the eigenvalue 1, i.e.

$$cu = T(cu) = cT(u) = c\lambda u,$$

hence $\lambda = 1$, which is a contradiction. Now, with respect to that order basis, the matrix presentation of $T$ is $(\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix})$ which is diagonal.
Of course, assuming standard spectral theory of finite dimensional vector spaces, the problem becomes much easier. The assumptions give us that the characteristic polynomial of $T$ is $\chi_T(x) = (x - 1)(x - \lambda)$, for some $\lambda \in \mathbb{R}$. If $\lambda \neq 1$ the characteristic polynomial splits into distinct linear factors, hence the linear operator is diagonal with respect to a basis of eigenvectors.

\begin{proof}
\end{proof}

**Problem 9.** Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose derivative satisfies the inequality

$$|g'(x)| \leq M \text{ for all } x \in \mathbb{R}.$$ 

Show that if $\varepsilon > 0$ is small enough, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + \varepsilon g(x)$ is one-to-one and onto.

**Solution.** Observe that for $\varepsilon$ small enough, the derivative of $f$ is close enough to 1:

$$|f'(x) - 1| = |(x + \varepsilon g(x))' - 1| = |1 - \varepsilon g'(x) - 1| = \varepsilon |g'(x)| \leq \varepsilon M,$$

Hence, if $\varepsilon < \frac{1}{M}$, $|f'(x) - 1| < 1$. In particular, for all $x \in \mathbb{R}$, $f'(x) > 0$. As a result, $f$ is strictly increasing on the whole real line, which also means that it is one-to-one.

Now, to verify that $f$ is onto, we have to verify that

$$\lim_{x \to -\infty} f(x) = -\infty, \text{ and } \lim_{x \to +\infty} f(x) = +\infty.$$ 

For convenience, let $\varepsilon \leq \frac{1}{2M}$. The previous computation shows that $|f'(x) - 1| \leq \frac{1}{2}$, and hence $f'(x) \geq \frac{1}{2}$.

- For any $x > 0$, the mean value theorem for $f$ on the interval $[0, x]$, says that there is a point $\xi_x \in (0, x)$, such that $f(x) - f(0) = f'(\xi_x)(x - 0)$. This implies that $f(x) \geq \frac{1}{2} x + f(0)$. Clearly

$$\lim_{x \to +\infty} (\frac{1}{2} x + f(0)) = +\infty, \text{ so } \lim_{x \to +\infty} f(x) = +\infty.$$ 

- Similarly, for any $x < 0$, the mean value theorem for $f$ on the interval $[x, 0]$, says that there is a point $\xi_x \in (x, 0)$, such that $f(x) - f(0) = f'(\xi_x)(x - 0)$. This implies that $f(x) \leq \frac{1}{2} x + f(0)$ (recall, $x < 0$). Clearly

$$\lim_{x \to -\infty} (\frac{1}{2} x + f(0)) = -\infty, \text{ so } \lim_{x \to -\infty} f(x) = -\infty.$$ 

\end{proof}

**Problem 10.** Let $G$ be a group of order 155.

(a) Show that $G$ must have a non-trivial proper normal subgroup.

(b) Suppose that $G$ (still of order 155) is abelian. Either prove that $G$ is cyclic or give a counterexample.

**Solution.** Observe that the prime factorization of 155 is 155 = 5 · 31.

- The Sylow theorem asserts that the number of Sylow 31-subgroups of $G$, $n_{31}$, has to divide 5 and also satisfy $n_{31} \equiv 1 \mod 31$. It is clear that the only positive integer satisfying both conditions is $n_{31} = 1$. Again, by the Sylow theorem, the unique Sylow 31-subgroup $S_{31}$ is normal in $G$. Since the order of that subgroup is 31, it is proper and non-trivial.

- Let $x, y \in G$ elements of order 5 and 31 respectively (those exist by Cauchy/Sylow theorem). Since the group is abelian, for any integer $n$, $(xy)^n = x^n y^n$. This implies that the order of $xy \in G$ has to be a multiple of both 5 and 31. Since those numbers are coprime, the order of $xy$ is 155, and hence $xy$ generates $G$, i.e. $G$ is cyclic.

Let’s give a variation of the above solution, which avoids using Cauchy’s theorem (but is essentially the argument to prove it in the abelian case). Let $x \in G$ non-trivial. If $x$ does not generate $G$, by Lagrange’s theorem $x$ has order 5 or 31, let’s assume it’s 5. Since $G$ is abelian, the cyclic group $(x) := H \leq G$ is normal. The quotient $G/H$ has order 31, so any non-trivial element $yH$ has order 31, i.e. $(yH)^{31} = y^{31} H = H$. This also shows that the representative $y$ has order at least 31 (if $k < 31$ is such that $y^k = e$, then $(yH)^k = y^k H = eH = H$, which contradicts the order of $yH$). If $y$ has order
larger than 31, \(y\) needs to have order 155 and hence generates \(G\), otherwise \(y\) has order 31, and \(xy\) generates \(G\) by the above argument.

Finally, if we are happy to rely on the classification of finite abelian groups, we know that there are only two possible ones that we can write down: \(\mathbb{Z}/155\mathbb{Z}\) and \(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/31\mathbb{Z}\). The former is clearly cyclic, so we would just need to verify that the latter is isomorphic to the former. Now this is easy, since the elements \(x, y\), which we spent time seeking above, are now given to us by the (direct) product decomposition:

\[
(x, y) = ([1], [0]) \quad y = ([0], [1]).
\]

\[\square\]

**Problem 11.** Let \(\Omega\) be a connected open set in the plane \(\mathbb{R}^2\) and let \(f(x, y)\) be a \(C^\infty\)-real-valued function with the property that \(\text{grad}(f) = 0\) at every point of \(\Omega\). Prove that \(f\) is constant.

**Solution.** We will first prove that \(f\) is constant on any open disc contained in \(\Omega\). Let \(D \subset \Omega\) be an open disc. Then for any \(y_0 \in \mathbb{R}\) such that the line \(y = y_0\) intersects the disc (in an open interval \(I \times \{y_0\}\)), we can define the function \(h_{y_0}(x) = f(x, y_0)\) for \(x \in I\). Then

\[
h'_{y_0}(x) = \frac{\partial f}{\partial x}(x, y_0) = 0
\]

Hence \(h_{y_0}\) is constant. This implies that \(f\) is constant on every horizontal chord of the disc. Similarly, we use that computation for the vertical diameter of the disc for the function \(v_{x_0}(y) = f(x_0, y)\)

\[
v'_{x_0}(y) = \frac{\partial f}{\partial y}(x_0, y) = 0,
\]

we get that \(f\) is constant on the vertical diameter. Since that diameter intersects all horizontal chords, \(f\) is constant on the disc. Pictorially:

![Diagram](image)

Now we will use the fact that \(\Omega\) is open and connected to show that \(f\) is constant on \(\Omega\). Pick a point \((x_0, y_0) \in \Omega\) and consider the value \(c = f(x_0, y_0)\).

- Since \(f\) is continuous, \(\{(x, y) \in \Omega : f(x, y) = c\} = f^{-1}(c)\) is a closed subset of \(\Omega\).
- Let \((x_1, y_1) \in f^{-1}(c)\). Since \(\Omega\) is open, there is an open disc \(D \subset \Omega\) containing \((x_1, y_1)\). By our previous argument \(f\) is constant on \(D\), and hence \(D \subset f^{-1}(c)\). As a result, \(f^{-1}(c)\) is an open subset of \(\Omega\).

But \(\Omega\) is connected, and since \(f^{-1}(c)\) is non-empty (it contains \((x_0, y_0)\)), it has to be the entire domain \(\Omega\), i.e. \(f\) is constant on \(\Omega\).

Observe that the argument above shows that if \(\text{grad}(f) = 0\), then \(f\) is constant on paths parallel to the axes. So it would suffice to show that if \(\Omega\) is an open connected set, then any two points can be connected via a path which is piecewise parallel to the axes. This is easy to demonstrate. Fix a point \(p \in \Omega\) and let \(U\) be the subset of \(\Omega\) consisting of points that can be connected to \(p\) via a path piecewise parallel to the axes.

- Since every point on a disc can be connected via such a path to the center of the disc, \(U\) is open.
- If \(\{x_n\}\) is a sequence in \(U\) converging to a point \(q \in \Omega\), since \(\Omega\) is open, \(q\) is contained in a disc, inside \(\Omega\). We know that the sequence \(\{x_n\}\) eventually enters the disc, since it converges to \(q\). Then we can connect \(p\) via such a path to a term of the sequence in the disc, and then connect that term to \(q\), i.e. \(q \in U\) and hence \(U\) is closed.
• \( p \in U \), i.e. \( U \neq \emptyset \).

As a result \( U = \Omega \), and we proved our claim.

Another nice solution uses the fundamental theorem of calculus for path integrals. If \( \gamma \) is a path from \( p \in \Omega \) to \( q \in \Omega \), then
\[
\int_{\gamma} \nabla(f) = f(q) - f(p).
\]
Since \( \nabla(f) = 0 \), the integral is 0 (regardless the path). As a result \( f \) is constant on every path component. Since \( \Omega \subset \mathbb{R}^2 \) is open and connected, it is also path connected. Hence \( f \) is constant on \( \Omega \).

\( \square \)

**Problem 12.** Let \( V_0, V_1, V_2 \) be subspaces of a real vector space \( V \), with \( V_0 \) a proper subspace of \( V_1 \) and of \( V_2 \). Let \( S : V_1 \to V_0 \) and \( T : V_0 \to V_2 \) be linear transformations.

(a) If \( V \) is finite dimensional, show that \( T \circ S : V_1 \to V_2 \) is neither injective nor surjective.

(b) Does the same conclusion necessarily hold if \( V \) is infinite dimensional? Give either a proof or a counterexample.

**Solution.** (a) This follows from a simple set theoretic observation:

**Claim:** Consider the functions between sets \( f : X \to Y \), \( g : Y \to Z \), and their composition \( g \circ f : X \to Z \). Then:

(i) If \( g \circ f \) is one-to-one, then so is \( f \).

(ii) If \( g \circ f \) is onto, then so is \( g \).

**Indeed:** (i) Let \( a, b \in X \) such that \( f(a) = f(b) \). Since \( g \) is a function, we have that \( g \circ f(a) = g \circ f(b) \). But \( g \circ f \) is one-to-one. Hence \( a = b \) and this shows that \( f \) is one-to-one.

(ii) We know that \( f(X) \subseteq Y \). This implies that \( g(f(X)) \subseteq g(Y) \subseteq Z \). But \( g \circ f \) is onto, hence \( g(f(X)) = Z \), which means that \( g(Y) = Z \), i.e. \( g \) is onto.

Returning to the problem, assume now that \( T \circ S \) is injective. This implies that \( S : V_1 \to V_0 \) is injective. Since subspaces of finite dimensional vector spaces are also finite dimensional, this implies that \( \dim V_1 \leq \dim V_0 \), which is impossible, since \( V_0 \) is a proper subspace of \( V_1 \) and hence \( \dim V_0 < \dim V_1 \).

Similarly, assume that \( T \circ S \) is surjective. This implies that \( T : V_0 \to V_1 \) is surjective, adn hence \( \dim V_0 \geq \dim V_2 \), which is again impossible, since \( V_0 \) is a proper subspace of \( V_2 \).

(b) If we do not assume that our vector spaces are finite dimensional, we can actually construct an example, such that \( T \circ S \) is an isomorphism (hence both injective and surjective). Let \( V_1 = V_2 = \mathbb{R}[x] \) the real vector space of polynomials over \( \mathbb{R} \), and \( V_0 = \{ f(x) \in \mathbb{R}[x] : f(0) = 0 \} \) the real subspace of polynomials with zero constant coefficient (or equivalently, the polynomials divisible by \( x \)). Consider the linear operators

\[
S : \mathbb{R}[x] \to V_0, \quad S(f(x)) = mx(f(x)) = xf(x)
\]
\[
T : V_0 \to \mathbb{R}[x], \quad T(f(x)) = \frac{d}{dx}(f(x)) = f'(x)
\]

Now it is easy to verify that \( T \circ S : \mathbb{R}[x] \to \mathbb{R}[x] \) is an isomorphism, since it just rescales the basic elements \( \{1, x, x^2 \ldots \} \):

\[
\begin{array}{ccc}
\mathbb{R}[x] & \xrightarrow{x^k \mapsto (k+1)x^k} & \mathbb{R}[x] \\
\downarrow \hspace{0.5cm} m_x & & \downarrow \hspace{0.5cm} \frac{d}{dx} \\
V_0 & & \end{array}
\]

A rescaled version of this example is the following: Let \( V_1 = V_2 = \mathbb{R}^N \) the vector space of sequences of real numbers, and \( V_0 \) the subspace of sequences whose first term is 0. If \( S : \mathbb{R}^N \to V_0 \) is the left unilateral shift, acting on the standard basis as \( S(e_i) = e_{i+1} \), where \( e_i \) is the sequence

\[
e_i(n) = \begin{cases} 
1, & \text{if } i = n \\
0, & \text{otherwise}
\end{cases}
\]
and $T : V_0 \to \mathbb{R}^N$ the right shift operator defined on the basis as $T(e_i) = e_{i-1}$, then $T \circ S = \text{id}_{\mathbb{R}^N}$. 

\qed