RIEMANNIAN GEOMETRY OF THE CURVATURE TENSOR

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ABSTRACT

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The curvature tensor is the most important isometry invariant of a Riemannian metric. We study several related conditions on the curvature tensor to obtain topological and geometrical restrictions. The first condition is the that the kernel of the curvature tensor has codimension either two or three. In which case, we conclude that positive curvature can only occur on topologically trivial manifolds (for arbitrary dimension when the kernel is codimension two and only in dimension 4 for codimension three kernel). In the last half, we study the three dimensional manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$. We obtain new examples of these, show that the fundamental group is free under basic assumptions, and give more explicit descriptions of the general case of these metrics.

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CHAPTER 1 : Introduction

We will study three classes of manifolds, each with a specific condition on their curvature. In each case, we get a topological constraint restricting the spaces which can have these metrics. The first class will be the so-called *conullity 2* manifolds in arbitrary dimensions. The second class will be the so-called *conullity 3* manifolds in dimension 4, which have similarities to the *conullity 3* manifolds. The third class are 3-manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$. This class is the special case of conullity 2 manifolds which are also *curvature homogeneous*.

In all three of these classes, we focus on the global geometry of such manifolds. Previous work on these is focused on local statements. In Chapter 2, we find that the only way to have positive curvature is to be locally reducible or to be topologically trivial, see Theorem 6. In Chapter 3, we study conullity 3 manifolds and find, analogously to conullity 2, that the only way to have positive curvature is to be locally reducible or to be topologically trivial, see Theorem 12. We show a finite volume splitting result analogous to one in [FZ16], see Theorem 11. In Chapter 4, we study the manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$ and determine a wide class of locally irreducible examples which have qualitatively new behavior compared to previous examples, see Theorem 24. We also determine that the fundamental group is free if the manifold is (globally) irreducible, see Theorem 35.

All three of these cases consider the nullity of the curvature tensor, which we define now and was first considered in [CK52]. Let (M^n, g) be a Riemannian manifold with curvature tensor R. Define the distribution

$$\ker R_p := \{X \in T_pM : R(X,Y)Z = 0 \text{ for all } Y, Z \in T_pM\}.$$

We call M^n *conullity 2* if at every point $p \in M$, $\ker R_p$ has dimension $n-2$. In general, we call M^n *conullity k* if at every point $p \in M$, $\ker R_p$ has codimension k, and we say *conullity at most k* if the codimension of $\ker R$ is at most k. We will study conullity 2 or 3 manifolds. The simplest example of these is $M = \Sigma \times \mathbb{R}^{n-2}$ with the product metric and Σ^2 any surface. This has conullity at most 2 and has conullity 2 if Σ has nowhere zero Gaussian curvature.

The study of curvature nullity goes back to a conjecture in [Nom68]. A *locally symmetric* metric is one where $\nabla R = 0$. A more general condition is that of a *semi-symmetric metric*: where, for all vector fields X, Y,

$$R(X,Y) \cdot R = 0. \tag{1.1}$$

Here $R(X,Y)$ is acting as a derivation on R, i.e. for all X, Y, U, V,

$$[R(X,Y), R(U,V)] - R(R(X,Y)U, V) - R(U, R(X,Y)V) = 0. \tag{1.2}$$

Cartan showed that all symmetric spaces satisfy (1.1). This is equivalent to the condition that for each point $p \in M$, R_p is the same, thought of algebraically as a tensor on T_pM, as the curvature tensor of some symmetric space. For example, $\Sigma \times \mathbb{R}^{n-2}$ with the product metric has the same tensor as $\Sigma(K_p) \times \mathbb{R}^{n-2}$ for $\Sigma(K_p)$ the symmetric surface of constant curvature K_p, with K_p the Gauss curvature of Σ at a point p.

Nomizu conjectured that all complete, irreducible semi-symmetric spaces of dimension at least 3 are locally symmetric [Nom68]. However, the first counter-example was found in [Tak72]. That example is the graph in $\mathbb{R}^4$ of

$$x_4 = \frac{\left(x_1^2 - x_2^2\right) x_3 - 2x_1x_2}{2\left(x_3^2 + 1\right)} \tag{1.3}$$

which is also the first example of an (irreducible) conullity 2 manifold.

Szabó proved the connection to conullity at most 2 manifolds all semi-symmetric, locally irreducible manifolds are of one of the following types: “trivial” examples which are either surfaces or locally symmetric, “exceptional” which are cones of elliptic, hyperbolic, Euclidean or Kählerian type, or“typical” which are Riemannian spaces foliated by Euclidean leaves of codimension 2 [Sza82, Sza85]. These "typical" manifolds are conullity at most 2 manifolds. Moreover the "exceptional" maniflds are necessarily non-complete, motivating the study of conullity at most 2. In [Sza84], the hypersurface case is examined thoroughly.

Conullity at most 2 arises in various other contexts. The classic Beez-Killing theorem says that any locally deformable hypersurface in $\mathbb{R}^n$ has conullity at most 2. The first examples of Riemannian manifolds with geometric rank one were Gromov's 3-dimensional graph manifolds which have metrics with conullity at most 2 [Gro78].

Another class of manifolds with conullity at most 2 are graph manifolds and in particular are easy to see that they are not globally reducible. In [FZ16], conditions are given for when a finite volume conullity at most 2 space is a graph manifold. Moreover, they prove that Nomizu's conjecture is true for finite volume manifolds. For the simplest graph manifold, start with a disk cross a circle, $B_1 \times S^1$ which has boundary $S^1 \times S^1$. Put a product metric on this so that the B_1 has a flat neck near its boundary. This can then be glued to another copy of it but with the boundary components swapped so that there is no global product structure. This is clearly conullity at most 2 (and conullity 2 near the boundary of B_1) since it is locally a product metric. In [FZ17], graph manifolds with nonnegative curvature are studied, contrasting with nonnegative curvature and conullity exactly 2 case in Chapter 2.

A more qualitative motivation is that the symmetries of R rule out $\dim \ker R_p = n - 1$ and so conullity 2 is "as close to trivial as R can get".

We now state the main result of Chapter 2, Theorem 6.

Theorem 1. *Suppose that for $n > 2$, M^n is complete, has conullity 2 and $\sec \geq 0$, and its universal cover is irreducible. Then M^n is diffeomorphic to $\mathbb{R}^n$.*

There is no known global example of a manifold satisfying all of these conditions, though local examples exist.

In Chapter 3, Theorem 11 generalizes a result of [FZ16] to work in conullity at most 3.

Theorem. *Let M^4 be a complete, finite volume, conullity at most 3 Riemannian manifold. Let V be a connected open subset of M on which the nullity leaves are complete and* $\dim \ker R = n - 3$*. Then the universal cover of V splits isometrically as $\tilde{D} \times \mathbb{R}$ where D is a maximal leaf of* $\ker R$ *in V.*

This is then used to prove Theorem 12 which is analogous to the conullity 2 result.

Theorem. *Suppose that M is a complete 4-dimensional Riemannian manifold has conullity 3 curvature,* $\sec \geq 0$*, and its universal cover is irreducible. Then M is diffeomorphic to $\mathbb{R}^4$.*

As in the conullity 2 result, there is no known global example of a manifold satisfying all of these conditions, though local examples.

The third chapter studies 3-manifolds whose Ricci eigenvalues are constants $(\lambda, \lambda, 0)$. This one case in the study of curvature homogeneous manifolds, introduced in [Sin60].

Definition 2. A manifold (M, g) is called *curvature homogeneous* if for any two points $p, q \in M$, there exists a linear isometry $f : T_pM \to T_qM$ such that $f^*(R_q) = R_p$.

We think of curvature homogeneous manifolds as being ones where the curvature tensor is "the same" at each point, up to some orthogonal map of the tangent spaces. Clearly any homogeneous manifold is also curvature homogeneous. Moreover, in dimension three, curvature homogeneity is the same as having constant Ricci eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ at all points since the Ricci tensor determines the curvature tensor.

A classical question asked by Singer [Sin60] is the following.

Question 3 (Singer)**.** *Are all curvature homogeneous manifolds locally homogeneous?*

The study of this question was motivated by the result in [Sin60] that if the curvature tensor and enough of its covariant derivatives match at each point, then M is locally homogeneous. The following conjecture provides a similar motivation [BKV96].

Conjecture 4 (Gromov)**.** *Fix a compact manifold M and a curvature tensor R. Then the space of curvature homogeneous metrics on M (up to isometry) which have curvature tensor*

R is finite dimensional.

In the case where M is non-compact, the example below in (1.4) and [SW15] shows that there may be an infinite dimensional moduli spaces. If the curvature tensor is that of an irreducible symmetric space, then the space is symmetric [TV86]. Curvature homogeneous manifolds have also been studied in the case cohomogeneity one actions, i.e. admitting an isometry group with a codimension one orbit [Ver97, Tsu88].

We now consider just the case of curvature homogeneity in dimension 3. All 3-manifolds with Ricci eigenvalues $(\lambda, \lambda, 0)$ are curvature homogeneous but most are not homogeneous. The first such example is from [Sek75] and has the metric of the form

$$g = p(x,u)^2 dx^2 + (du - v\, dx)^2 + (dv + u\, dx)^2 \tag{1.4}$$

where either $p(x,u) = A(x)\cosh u + B(x)\sinh u$ (corresponding to eigenvalues $(-1,-1,0)$), or $p(x,u) = A(x)\cos u + B(x)\sin u$ (corresponding to $(1,1,0)$). However, only the case with $(1,1,0)$ eigenvalues does not give a complete metric.

In dimension 3, classifying curvature homogeneous manifolds breaks into three cases.

(a) $\lambda_1 = \lambda_2 = \lambda_3$, which implies constant sectional curvature,

(b) $\lambda_1, \lambda_2, \lambda_3$ all distinct, or

(c) $\lambda_1 = \lambda_2 \neq \lambda_3$.

The first case is trivial. All three cases are well studied locally [KP94, BKV96, Kow93].

We restrict our attention to the subcase of (b) where $\lambda_3 = 0$. This case, with eigenvalues $(\lambda, \lambda, 0)$ is of particular interest as it is the only case where the metric can have a local product structure in some regions but not in others.

The relation between this condition and the previous ones studied is that if M^3 has Ricci eigenvalue $(\lambda, \lambda, 0)$, then M is also conullity 2.

We may assume that, up to scaling, $\lambda = -1$ since otherwise, the universal cover of such a manifold splits as $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{E}^2 \times \mathbb{R}$, see Lemma 15.

Manifolds with constant Ricci eigenvalues $(\lambda, \lambda, 0)$ are also *constant vector curvature 0* manifolds, i.e. for every vector $X \in T_pM$ there exists $Y \in T_pM$ such that the sectional curvature $\sec(X, Y) = 0$. This condition gives similar finite-volume splitting results to Theorem 11, see [SW14, SW17]. Under a stronger curvature assumption, known as higher rank, there is a splitting for all 3-manifolds [BS18].

We now state the main results that we prove in Chapter 4. First, we give in Theorem 24 a construction of many examples of locally irreducible, simply connected manifolds with Ricci eigenvalues $(-1, -1, 0)$.

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and $\gamma : \mathbb{R} \to \mathbb{H}^2$ be a C^1 arc-length parametrized curve in $\mathbb{H}^2$ such that the geodesics orthogonal to γ are disjoint and cover $\mathbb{H}^2$. Define $S \subset \mathbb{R}$ to be the set of x such that $\gamma(x)$ is locally a smooth curve and define $h(x) = \nabla_{\gamma'}\gamma'$ on S. Suppose further that $f(x)$ satisfies that*

$$f^{(k)}(x)h^{(\ell_1)}(x)\cdots h^{(\ell_m)}(x) \to 0 \tag{1.5}$$

as $x \to x_0 \notin S$, for any $k, m, \ell_1, \ldots, \ell_m \geq 0$.

Then there exists a complete metric g on $\mathbb{H}^2 \times \mathbb{R}$ with Ricci eigenvalues $(-1, -1, 0)$.

Next, we show in Theorem 29 that under an assumption, all such manifolds are of the form of these examples.

Theorem. *Suppose M is complete and simply connected and has Ricci eigenvalues $(-1, -1, 0)$. If M is everywhere locally irreducible and $\mathcal{F}$ is smooth on M, then M has smooth coordinates*

(x,u,v) such that

$$g = (\cosh u - h(x)\sinh u)^2 dx^2 + (du - f(x)v\;dx)^2 + 9dv + f(x)u\;dx)^2 \tag{1.6}$$

for some smooth functions $f, h : \mathbb{R} \to \mathbb{R}$ with $|h| \leq 1$.

Presumably this assumption that the foliation $\mathcal{F}$ is smooth should not be needed to know that locally irreducible, simply connected manifolds with Ricci eigenvalues $(-1,-1,0)$ are of the form in Theorem 24.

Finally, we classify in Theorem 35 the topology of non-simply connected manifolds with Ricci eigenvalues $(-1,-1,0)$under another technical condition.

Theorem. *Suppose that (M^3, g) is complete, has Ricci eigenvalues $(-1,-1,0)$ and has locally finite split regions. If its universal cover $\widetilde{M}$ is irreducible, then $\pi_1(M)$ is a free group. If M is locally irreducible everywhere, then $\pi_1(M)$ is either trivial or $\mathbb{Z}$.*

1.1. Preliminaries

It is well known that $\ker R$ has complete, totally geodesic leaves on the open subset where $\dim\ker R$ is minimal [Mal72]. (For a conullity k manifold, this subset is all of M, for conullity at most k it is the set where $\dim\ker R$ is exactly $n-k$.) Moreover, these leaves are flat since they are spanned by $\ker R$. Any geodesic contained in a leaf of $\ker R$ is called a *nullity geodesic*, which exist for all $T \in \ker R$. Since $\ker R$ has totally geodesic leaves, the orthogonal distribution $(\ker R)^\perp$ is parallel along the leaves of $\ker R$.

Following the conventions in [FZ16], we define the *splitting tensor* C_T for any $T \in \ker R$ by

$$(C_T)_p(X) = -(\nabla_X T)^{\ker R_p^\perp} \tag{1.7}$$

where $(\cdot)^{\ker R_p^\perp}$ means the orthogonal projection onto $(\ker R_p)^\perp$.

Moreover, from [FZ16], for vector fields $U, S \in \ker R$,

$$\begin{aligned} C_{\nabla_U S} X &= -(\nabla_U \nabla_X S)^{\ker R^\perp} - (\nabla_{[X,U]} S)^{\ker R^\perp} \\ &= (\nabla_U C_S) X + C_S(\nabla_U X) - C_S([U,X]^{\ker R^\perp}) \\ &= (\nabla_U C_S) X + C_S(\nabla_X U) \\ &= (\nabla_U C_S) X - C_S C_U X \end{aligned}$$

Hence, we obtain a Ricatti-type equation,

$$\nabla_U C_S = C_{\nabla_U S} + C_S C_U. \tag{1.8}$$

Along a nullity geodesic $\gamma(t)$ with tangent vector $T \in \ker R$, we can choose a parallel basis $\{e_1, e_2\}$ of $\ker R^\perp$. Then C_T written in this basis is a matrix $C(t)$ along $\gamma(t)$ satisfying

$$C'(t) = C^2 \tag{1.9}$$

and hence has solutions $C(t) = C_0(I - tC_0)^{-1}$ for some $C_0 = C(0)$ matrix. Hence C_T must have no non-zero real eigenvalues.

When M is conullity at most 2, then C_T is a 2×2 matrix and hence either is nilpotent or has two non-zero complex eigenvalues. When M is conullity at most 3, then C_T is 3×3 matrix and hence always has a zero eigenvalues. Moreover, C_T then is again either nilpotent or has two non-zero complex eigenvalues. In either conullity 2 or 3, these two cases lead to qualitatively different behavior.

We will make use of the following de Rham-type splitting result, from [FZ16] and done in dimension 3 in [AMT15].

Proposition 5. *Let M be a complete Riemannian manifold of conullity at least 2 and $V \subset M$ a connected open subset on the nullity geodesics are complete. If $C_T \equiv 0$ on U for all $T \in \ker R$, then the universal cover of V is isometric to $\tilde{D} \times \mathbb{R}^{n-2}$ where $\tilde{D}$ is a simply* *connected surface.*

CHAPTER 2 : Conullity 2

2.1. Nonnegative Curvature

Our main result on conullity 2 manifolds is the following.

Theorem 6. *Suppose that M^n has $n > 2$, conullity 2,* $\sec \geq 0$*, and its universal cover does not split isometrically. Then $\widetilde{M}^n$ is diffeomorphic to $\mathbb{R}^n$.*

Since $\sec \geq 0$, M has a soul $S \subset M$ by the soul theorem [CG72, Per94]. Each point $p \in M$ there is an orthonormal basis of the form $\{e_1, e_2, T_1, \ldots, T_{n-2}\}$ of T_pM with $R(T_j, \cdot) = 0$ and $\sec(e_1, e_2) = \text{Scal}$. The case where $S = M$, i.e. M is compact, is covered in [FZ16], showing that $\widetilde{M}$ is isometric to $\widetilde{D} \times \mathbb{R}^{n-2}$ for a positively curved surface D. In [FZ16] (as well as in [AMT15] for dimension 3), the following result covers the finite-volume case without a curvature assumption.

Proposition 7. *If a complete manifold M has conullity at most 2 and is finite volume, then its universal cover $\widetilde{M}$ splits isometrically as $\Sigma \times \mathbb{R}^{n-2}$ for some complete surface Σ.*

We first show how the nullity vectors of T_pM can fit in with the Soul.

Lemma 8. *If $T \in \ker R_p$ at a point $p \in S$, then the orthogonal projections $T^S \in T_pS$ and $T^N \in T_pS^\perp$ are also in* $\ker R_p$.

Proof. First observe that since $T = T^S + T^N \in \ker R_p$, that

$$\left\langle R(T^S, X)Y, Z \right\rangle = -\left\langle R(T^N, X)Y, Z \right\rangle \tag{2.1}$$

for any X, Y, Z. Take a unit vector $e \in T_pM$ orthogonal to T and write e^S and e^N as its projections. Then

$$\begin{aligned}\sec(e, T^N) &= \langle R(e^N, T^N)T^N, e^N\rangle + \langle R(e^S, T^N)T^N, e^N\rangle \\ &\quad + \langle R(e^N, T^N)T^N, e^S\rangle + \langle R(e^S, T^N)T^N, e^S\rangle \\ &= \underbrace{\langle R(e^N, T^N)T^N, e^N\rangle}_{(a)} + \underbrace{2\langle R(e^S, T^N)T^N, e^N\rangle}_{(b)} + \underbrace{\langle R(e^S, T^N)T^N, e^S\rangle}_{(c)}\end{aligned}$$

This last term (c) is 0 since it is the sectional curvature of one of the flat strips from the proof of the Soul Conjecture [Per94]. The first term (a) can be written using (2.1) as

$$(a) = -\langle R(e^N, T^S)T^N, e^N\rangle = \langle R(e^N, T^S)T^S, e^N\rangle = 0$$

to see this as again the curvature of a flat strip and hence zero.

For (b), we use the fact that the flat strip from e^S and T^N is totally geodesic, and so $R(e^S, T^N)T^N$ is in the span of $\{e^S, T^N\}$ and hence $(b) = 0$.

This shows that T^N has $\sec(T^N, \cdot) = 0$. Using 2.1 twice then also gives that $\sec(T^S, \cdot) = 0$.

This is sufficient to show that T^N and T^S are in $\ker R_p$, as any $X \notin \ker R_p$ has some non-zero sectional curvature. Specifically, we can write $X = X^T + X^e$ with $X^T \in \ker R_p$ and $0 \neq X^e \in \ker R_p^{\perp}$ and there exists $e_2 \perp X^e$ in $\ker R_p^{\perp}$. Then $\sec(X, e_2) = \sec(X^e, e_2) = \mathrm{Scal}$.

□

Our next lemma then tells us how to make a basis of the tangent space at a point of the soul that fits nicely with both the soul structure and the conullity 2 structure.

Lemma 9. *For $p \in S$, there exists an orthonormal basis $B = \{e_1, e_2, T_1, \ldots, T_{n-2}\}$ of T_pM so that each basis vector $v \in B$ is either in T_pS or in $T_pS^{\perp} \subset T_pM$ and B satisfies the relations*

$$R(T_j, \cdot) = 0, \quad \sec(e_1, e_2) = \mathrm{Scal}\,.$$

Moreover, e_1 and e_2 are either both in T_pS or both in $T_pS^\perp$.

Proof. Pick any basis $S_1, \dots, S_{n-2}$ of $\ker R_p$. Then $S_1^N, \dots, S_{n-2}^N, S_1^S, \dots, S_{n-2}^S$ also spans $\ker R_p$, so take a subset of this that is a basis and call it $T_1, \dots, T_{n-2}$. Now chose e_1, e_2 perpendicular to the span of $T_1, \dots, T_{n-2}$ with each e_i either in T_pS or $T_pS^\perp$. Then e_1, e_2 span $\ker R_p^\perp$ and $\{e_1, e_2, T_1, \dots, T_{n-2}\}$ is our desired basis.

Moreover, note that if $e_1 \in T_pS$ and $e_2 \in T_pS^\perp$, then there is a flat strip spanned by e_1 and e_2, so $\sec(e_1, e_2) = 0$, which is a contradiction with the assumption that $\mathrm{Scal} > 0$ everywhere. So e_1 and e_2 must both be in T_pS or both be in $T_pS^\perp$. □

Further note that since $\ker R_p$ and $\ker R_p^\perp$ are smooth distributions and the soul S is connected, that if e_1 and e_2 are in T_pS at one point, they must be so at every point. So there are now two cases, case (A) where $e_1, e_2 \in T_pS$ for all $p \in S$ and case (B) where $e_1, e_2 \in T_pS^\perp$ for all $p \in S$. The first case is where the soul contains all the curvature and the second case is where the soul is flat.

In this first case, the soul S of M is a compact manifold with conullity 2 at each point. So we may apply the splitting result 7 to the soul to get that $\widetilde{S}$ is isometric to $\widetilde{D} \times \mathbb{R}^{m-2}$ where m is the dimension of S. Here $\widetilde{D}$ is a simply connected, compact surface with $\mathrm{Scal} > 0$, and hence is diffeomorphic to $\mathbb{S}^2$.

Now we examine the splitting tensor at $p \in S$. If $T \in \ker R_p$ is any nullity vector, then C_T is defined by $C_T(X) = -(\nabla_X T)^{\ker R_p^\perp}$. If $T \in T_pS$, then $C_T = 0$ by the splitting of $\widetilde{S}$. Otherwise, assume that T is perpendicular to T_pS. For $X \in \ker R_p^\perp$, then the flat strip spanned by X and T is totally geodesic. Since C_T is a tensor, we can choose to consider extensions of X and T to vector fields contained in that flat strip. For these extensions, $\nabla_X T$ is in the span of X and T. Since $C_T(X) \in \ker R_p^\perp$, it must be perpendicular to T and hence X is an eigenvector of $C_T(X)$ with a real eigenvalue. The only possible such eigenvalue is 0. So $C_T = 0$ as well.

So all splitting tensors are zero on S. For any other point $p \in M$, $p = \exp_{p_0}(U)$ for some $p_0 \in S$ and $U \in T_pS^\perp$. Since $e_1, e_2 \in T_pS$, we know that $U \in \ker R_p$. By (1.8), $C_U \equiv 0$ along $\gamma(t) = \exp_{p_0}(tU)$ since $C_U = 0$ at $p_0 \in S$. For any $T \in \ker R_{p_0}$, we know that $C_T = 0$ at p_0. By (1.8) extending T parallel along γ, we get that

$$\nabla_U C_T = C_T C_U = 0.$$

Hence, $C_T \equiv 0$ along γ and in particular $C_T = 0$ at p. Since $\ker R$ is parallel along γ, $C_T = 0$ at p for all $T \in \ker R_p$.

So all splitting tensors are identically zero on M. By Proposition 5, we conclude that $\widetilde{M}$ splits isometrically as $\widetilde{D} \times \mathbb{R}^{n-2}$ with the Euclidean metric on $\mathbb{R}^{n-2}$.

We now consider the second case, where the soul is flat. We write this case as a lemma, which will also be used in our theorem on conullity 3 manifolds.

Lemma 10. *Suppose that M has a soul S of dimension at least one. Then if S is flat, $\widetilde{M}$ splits isometrically with a Euclidean factor.*

Proof. In this case, since S is flat, we know that its universal cover $\widetilde{S}$ is just flat $\mathbb{R}^m$.

Let $\nu(S)$ be the normal bundle of S in M. If $\pi : \widetilde{S} \to S$ is the universal covering of S, then $\widetilde{M}$ is diffeomorphic to the pullback bundle

$$\pi^*(\nu(S)) = \left\{ (\widetilde{p}, X) \middle| \, \widetilde{p} \in \widetilde{S}, \; X \in T_{\pi(\widetilde{p})}S^\perp \right\}.$$

We can see this since there is a covering map $\pi^*(\nu(S)) \to \nu(S)$ to the normal bundle of S given by $(\widetilde{p}, X) \mapsto (\pi(\widetilde{p}), X)$ and noting that $\pi^*(\nu(S))$ is simply connected since $\widetilde{S}$ is. That shows that $\pi^*(\nu(S))$ is the universal cover of $\nu(S)$, which is diffeomorphic to M. So $\widetilde{M}$ is diffeomorphic to $\pi^*(\nu(S))$, a vector bundle over Euclidean space $S = \mathbb{R}^m$. Hence $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$.

Moreover, suppose that $m > 0$ so that the soul S is not just a point. The fact that $\pi_1(S) = \pi_1(M)$ implies that $\widetilde{S}$ is embeddable in $\widetilde{M}$ (since distinct homotopy classes of paths in S are still distinct in M). Since S is totally geodesic and totally convex in M, so is $\widetilde{S}$ in $\widetilde{M}$ (since geodesics of $\widetilde{M}$ are just lifts of geodesics in M).

Now take a line L in $\widetilde{S} = \mathbb{R}^m$. Take any two points x, y on the line L. Then any minimizing geodesic from x to y must lie in $\widetilde{S}$, since $\widetilde{S}$ is totally convex, and the only such geodesics are just the line L itself. So L is a line in $\widetilde{M}$ in the sense of minimizing distance between any two points of it. Hence $\widetilde{M}$ splits isometrically as $N^{n-1} \times \mathbb{R}$. Here N^{n-1} has a soul with dimension $m-1$. This process can be repeated to get that $\widetilde{M} = N^{n-m} \times \mathbb{R}^m$ isometrically with flat $\mathbb{R}^m$ for some manifold N^{n-m} with soul a point, so N^{n-m} is diffeomorphic to $\mathbb{R}^{n-m}$. □

Combining this with case (A) proves the theorem.

2.2. Examples

The first examples of conullity 2 manifolds were from Takagi [Tak72] and Sekigawa [Sek75]. Examples deriving from Sekigawa's will be studied in extensive detail in Chapter 4. We now present several other explicit examples.

2.2.1. 2 Complex Eigenvalues and Complete

The following example is derived from the final example of Section 8 of [Kow96] with $U = \log(\log(x+1))$. The metric

$$g = \left(x^2 + \log(x+1)\right)\left(y^2 \log(x+1) + 1\right)(x+1)\, x\, dw^2 + x(dw\, dy + dy\, dw) \tag{2.2}$$

$$+ \frac{y^2 \log(x+1) + 1}{x(x+1)\left(\log(x+1)\right)^2} dx^2 + dy^2 \tag{2.3}$$

defined on (w, x, y) with $x > 0$ is a complete conullity at most 2 metric with 2 complex eigenvalues for its splitting tensor. Its scalar curvature is negative at all points. The vector field $\frac{\partial}{\partial y}$ is T, the nullity direction.

We can see the completeness of this metric by noting that $g(X,Y) \geq \bar{g}(X,Y)$ where $\bar{g}$ is defined by

$$\bar{g} = x^4 dw^2 + \frac{dx^2}{x(x+1)(\log(x+1))^2} + dy^2.$$

Hence completeness of $\bar{g}$ implies completeness of g and $\bar{g}$ we can see is complete by making the reparametrization by $\bar{x} = \log(\log(x+1))$ so

$$\bar{g} = (e^{e^{\bar{x}}} - 1)^4 dw^2 + d\bar{x}^2 + dy^2.$$

2.2.2. 2 Complex Eigenvalues and Positive Scalar Curvature

The following example is derived from the final example of Section 8 of [Kow96] with $U = x^{3/2}$. The conullity at most 2 metric

$$g = \left(x^2 + \frac{2}{3}\sqrt{x}\left(e^{x^{3/2}} y^2 + 1\right)\right) dw^2 + x(dw\,dy + dy\,dw) + \frac{3}{8}\frac{y^2 + e^{-x^{3/2}}}{\sqrt{x}} dx^2 + dy^2$$

defined for (w,x,y) with $x > 0$ has positive scalar curvature for small x (and arbitrary w,y). In particular, the scalar curvature K is

$$K = -\frac{\left(21x^{3/2} + 9x^3 - 2\right) e^{x^{3/2}}}{3x^{3/2}}.$$

The metric is non-complete since paths with w and y constant have finite length as $x \to \infty$ or $x \to 0$. The vector field $\frac{\partial}{\partial y}$ is T, our nullity direction. The splitting tensor C for this has two complex eigenvalues, when $K \neq 0$ (and is not defined for surface where $K = 0$). In particular C has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where

$$a = -\frac{y}{y^2 + e^{-x^{3/2}}}, \qquad b = \frac{e^{-\frac{1}{2}x^{3/2}}}{y^2 e^{x^{3/2}} + 1}. \tag{2.4}$$

The surface on which $\operatorname{tr} C = 0$ is $y = 0$ which is always orthogonal to the nullity direction.

CHAPTER 3 : Conullity 3

We have two main results for conullity 3 manifolds in dimension 4. The first is analogous of the finite volume splitting result 7 from [FZ16].

Theorem 11. *Let M^4 be a complete, finite volume, conullity at most 3 Riemannian manifold. Let V be a connected open subset of M on which the nullity leaves are complete and* $\dim \ker R = n - 3$. *Then the universal cover of V splits isometrically as $\tilde{D} \times \mathbb{R}$ where D is a maximal leaf of* $\ker R$ *in V.*

The second theorem is analogous to Theorem 6

Theorem 12. *If M is a 4-dimensional Riemannian manifold with conullity 3 curvature such that* $\sec \geq 0$, *then either the universal cover $\widetilde{M}$ splits isometrically with a Euclidean factor, or M is diffeomorphic to $\mathbb{R}^4$.*

A very similar splitting result is proved for arbitrary odd conullity under the different curvature assumption that the sectional curvature of all planes orthogonal to $\ker R$ are non-zero in [Ros67] and generalized in [Ros69].

We follow closely the strategy for conullity 2 manifolds done in [FZ16]. In particular, we make use of their result:

Proposition 13. *Let M be a complete Riemannian manifold, and $V \subset M$ a connected open subset which has the parallel rank k distribution* $\ker R$ *whose leaves are flat and complete. Then, the universal cover of V is isometric to $\tilde{D} \times \mathbb{R}^k$, where $\tilde{D}$ is the universal cover of a maximal leaf D of* $\ker R^{\perp}$. *Furthermore, the normal exponential map* $\exp^{\perp} : T^{\perp}D \to V$ *is an isometric covering map if $T^{\perp}D$ is equipped with the induced connection metric.*

Recall that in conullity at most 3, C is a 3×3 matrix in a parallel basis along γ. Hence $C(t)$ has at least one real eigenvalue, and therefore has 0 as an eigenvalue. The two possibilities are then that either C has two complex eigenvalues and one 0 eigenvalue or that C is nilpotent.

3.1. Finite Volume

We now prove Theorem 11.

Proof. Since M is 4 dimensional, $\ker R$ is one dimensional. Hence, passing to a double cover of V if necessary, we may pick a unit vector field $T \in \ker R$ on V. Moreover, $\nabla_T T = 0$ since $\ker R$ is totally geodesic, so T integrates to geodesics. Define $C := C_T$ on V. We will show that $C = 0$.

First, we look at the case where C has two non-zero complex eigenvalues and one zero eigenvalue. Then in an appropriate choice of parallel basis,

$$C(t) = \begin{pmatrix} A(t) & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

where A is a 2×2 matrix with 2 complex eigenvalues. By Equation 1.8, we have

$$\operatorname{tr} A(t) = \frac{\operatorname{tr} A_0 - 2t \det A_0}{1 - t \operatorname{tr} A_0 + t^2 \det A_0} \text{ and } \det A(t) = \frac{det A_0}{1 - t \operatorname{tr} A_0 + t^2 \det A_0} \tag{3.2}$$

Take $B \subset V$ a small compact neighborhood. Since $\det A(0) > 0$, there is some time t_0 so that $\operatorname{tr} C(t) = \operatorname{tr} A(t) \leq 0$ for all $t \geq t_0$. In the second case, where C is nilpotent, then $\operatorname{tr} C = 0$. In either case, $\operatorname{tr} C \geq 0$ for $t \geq t_0$ for some t_0.

Note that

$$\operatorname{div} T = \operatorname{tr} \nabla T = -\operatorname{tr} C. \tag{3.3}$$

Now define $B_t := \phi_{t+t_0}(B)$ where ϕ is the flow along T. Then

$$\frac{d}{dt} \text{vol } B_t = \int_B \frac{d}{dt} \phi^*_{t+t_0}(d\,vol) = \int \operatorname{div} T = -\int_B \operatorname{tr} C(t + t_0) \geq 0 \tag{3.4}$$

for all $t \geq 0$. Hence, the flow of T is volume non-decreasing and we get, by weak recurrence, a sequence of compact neighborhoods B_{n_k}, with $\{n_k\} \in \mathbb{N}$ an increasing sequence, so that $B_{n_k} \cap B_{n_0} \neq \emptyset$. This gives a sequence of points, $p_k := \phi_{t_0+n_k}(q_k) \in B_{n_k} \cap B_{n_0}$, with $q_k \in B$, with an accumulation point $p \in B_{n_0} \subset V$.

First consider $V' \subset V$, the open subset on which C has non-zero complex eigenvalues. By (1.8), V' is invariant under the flow ϕ_t of T. The sequence of points $p_k \to p$ and Equation 3.2 gives

$$\det A_{T(p)} = \lim_{k\to\infty} \det A_{T(p_k)} = \lim_{k\to\infty} \frac{\det A_{T(q_k)}}{1-(t_0-n_k)\operatorname{tr} A_{T(q_k)} + (t_0-n_k)^2 \det A_{T(q_k)}} = 0 \tag{3.5}$$

where again $A_{T(q_k)}$ is the 2×2 block of $C_{T(q_k)}$ with two non-zero complex eigenvalues. Therefore $A_{T(p)} = 0$ and so $C = 0$.

A concern here is that $\operatorname{tr} A_{T(q_k)}$ and $\det A_{T(q_k)}$ is not obviously continuous since it involved a choice of basis and each nullity geodesic will require it's own choice of basis. However, $\operatorname{tr} A_{T(q_k)}$ is just $\operatorname{tr} C$, and $\det A_{T(q_k)}$ can be written as

$$\det A_{T(q_k)} = \lambda_1\lambda_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{(\operatorname{tr} C)^2 - \operatorname{tr}(C^2)}{2} \tag{3.6}$$

for eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of C (with $\lambda_3 = 0$). So both expressions are actually invariant under change of bases and so are continuous.

Consider next the other case and define V^* to be the open subset of V on which C is nilpotent and non-zero. The previous case differs only slightly from the argument in conullity 2, but the nilpotent case requires significantly more computations than in conullity 2.

First, we will find vector fields on V^* giving an orthonormal basis. Observe that

$$l \dim\ker C^2 = \dim\ker C + 1 \tag{3.7}$$

since C is 3×3, nilpotent, and non-zero. Hence, define e_2 to be a unit vector field spanning

$\ker C^2 \cap (\ker C)^\perp$ on V^*, passing to a double cover of V if necessary. Then let e_1 be a unit vector field parallel to $C(e_2)$ and e_3 a unit vector field perpendicular to e_1 and e_2, passing to an orientable cover of V^* if necessary. This gives an orthonormal basis of vector fields $\{e_1, e_2, e_3\}$ on which we can write C as

$$C = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

Note that by this construction, a is non-zero at every point on V^*, though b and c possibly could be zero. Moreover, (1.8) shows that $\ker C$ and $\ker C^2$ are parallel along nullity geodesics. Hence, e_1, e_2, e_3 are parallel along nullity geodesics.

Then (1.8) gives

$$C(t) = \begin{pmatrix} 0 & a & c + tab \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

where a, b, c are independent of t.

Since the flow ϕ_t of T is volume preserving ($\operatorname{tr} C = 0$), the Poincaré recurrence theorem says that for almost all $p \in V^*$, there exists a sequence $t_n \to \infty$ with $\phi_{t_n}(p) \to p$. Hence $\langle C(t)e_3, e_1 \rangle = c + tab$ must be constant, not linear, and hence $b = 0$ when M is finite volume. Thus $\ker C(t)$ is 2 dimensional. This allows us to choose a better basis (again, in a cover of V^*), namely let e_2 be perpendicular to $\ker C$, e_1 parallel to Ce_2 and e_3 perpendicular to e_1, e_2. Then $\{e_1, e_2, e_3\}$ is an orthonormal basis so that

$$C = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

and C is constant in the T direction.

Now we do some computations in this basis. First, we denote the various Christoffel symbols by the Greek letters $\alpha, \beta, \gamma, \delta, \mu, \zeta, \eta, \xi, \nu$ which give smooth functions on M and satisfy

$$\nabla_{e_1} T = 0 \qquad \nabla_{e_2} T = -a e_1 \qquad \nabla_{e_3} T = 0 \qquad \nabla_T e_i = 0 \tag{3.11}$$

$$\nabla_{e_1} e_1 = \alpha e_2 + \beta e_3 \qquad \nabla_{e_2} e_1 = -\gamma e_2 + \xi e_3 + aT \qquad \nabla_{e_3} e_1 = \nu e_2 - \mu e_3 \tag{3.12}$$

$$\nabla_{e_1} e_2 = -\alpha e_1 + \eta e_3 \qquad \nabla_{e_2} e_2 = \gamma e_1 + \delta e_3 \qquad \nabla_{e_3} e_2 = -\nu e_1 - \zeta e_3 \tag{3.13}$$

$$\nabla_{e_1} e_3 = -\eta e_2 - \beta e_1 \qquad \nabla_{e_2} e_3 = -\xi e_1 - \delta e_2 \qquad \nabla_{e_3} e_3 = \mu e_1 + \zeta e_2 \tag{3.14}$$

We compute some curvature expressions in terms of these:

$$R(T, e_1)e_2 = -T(\alpha)e_1 + T(\eta)e_3 \tag{3.15}$$

$$R(T, e_1)e_3 = -T(\eta)e_2 - T(\beta)e_1 \tag{3.16}$$

$$R(T, e_2)e_1 = -(T(\gamma) + a\alpha)e_2 + (T(\xi) - a\beta)e_3 \tag{3.17}$$

$$R(T, e_2)e_3 = (-T(\xi) + a\beta)e_1 + (-T(\delta) + a\eta))e_2 \tag{3.18}$$

$$R(T, e_3)e_1 = T(\nu)e_2 - T(\mu)e_3 \tag{3.19}$$

$$R(T, e_3)e_2 = -T(\nu)e_1 - T(\zeta)e_3 \tag{3.20}$$

We know that all of these must be 0 since $T \in \ker R$, and hence $\alpha, \beta, \eta, \nu, \mu, \zeta$ are all constant in t. Moreover, $T(\gamma) = -a\alpha, T(\xi) = a\beta, T(\delta) = a\eta$ show that γ, δ, ξ all grow linearly in t. By Poincaré recurrence again, we must have that they actually are constant, hence $\alpha = \beta = \eta = 0$ and γ, ξ, δ are constant in t. In particular, all of the Christoffel symbols are constant along nullity geodesics, as is a.

Since a is constant along nullity geodesics we get that $T(e_1(a)) = [T, e_1](a) - e_1(T(a)) = 0(a) - e_1(0) = 0$ so $e_1(a)$ is also constant along nullity geodesics. We further compute that

$$T(e_2(a)) = e_2(T(a)) - [T, e_2](a) = ae_1(a) \tag{3.21}$$

Hence this means that $e_2(a)$ grows linearly along nullity geodesics. Poincaré recurrence again gives that $T(e_2(a)) = ae_1(a) = 0$, so $e_1(a) = 0$. Note that this argument works to show that $e_1(f) = 0$ for any f that is constant along nullity geodesics.

Some more curvature computations give (without assuming finite volume):

$$\langle R(e_2, e_3)e_3, T\rangle = \mu a \tag{3.22}$$

$$\langle R(e_3, e_2)e_2, T\rangle = \nu a \tag{3.23}$$

$$\langle R(e_2, e_1)e_1, T\rangle = -e_1(a) + \gamma a = \gamma a \tag{3.24}$$

Since $T \in \ker R$, we must again have all of these terms 0, hence $\mu = \nu = \gamma = 0$.

Computing a second Bianchi identity gives

$$0 = \nabla_{e_1} R(e_2, e_3)e_3 + \nabla_{e_2} R(e_3, e_1)e_3 + \nabla_{e_3} R(e_1, e_3)e_2 \tag{3.25}$$

$$= \left[e_1(e_3(\xi) - \delta\xi) + \zeta^2\xi\right] e_1 \tag{3.26}$$

$$+ \left[e_1(e_2(\zeta) + e_3(\delta) - \delta^2 - \zeta^2) + \delta\zeta\xi - e_3(\zeta\xi)\right] e_2 \tag{3.27}$$

$$+ \left[-\zeta^2\xi - e_3(\delta) + e_2(\zeta) - \delta^2 - \zeta^2\right] e_3 \tag{3.28}$$

In particular, $0 = e_1(e_3(\xi) - \delta\xi) + \zeta^2\xi$. Since $T(\xi) = 0$, $T(e_3(\xi)) = 0$ and so we get that $f := e_3(\xi) - \delta\xi$ is constant along nullity geodesics. By the argument that $e_1(f) = 0$ if $T(f) = 0$, we get that $e_1(e_3(\xi) - \delta\xi) = 0$. The second Bianchi identity then gives that $\zeta^2\xi = 0$, and in particular then $\zeta\xi = 0$.

In summary, all of $\alpha, \beta, \eta, \mu, \nu, \delta$ are zero and $\zeta\xi = 0$ as well. We use these to show that $R(e_1, \cdot)\cdot = 0$, which is a contradiction with the assumption that V^* has conullity exactly 3. Direct computation shows that, without any assumption of finite volume, that $R(e_1, \cdot)\cdot$ is determined by:

$$R(e_1,e_2)e_2 = \left[e_1(\gamma) - \delta\beta + e_2(\alpha) + \eta\xi + \alpha^2 + \eta\nu + \gamma^2 + \xi\nu\right] e_1 \tag{3.29}$$

$$+ \left[e_1(\delta) + \gamma\beta - e_2(\eta) + \alpha\xi + \alpha\eta + \eta\zeta + \gamma\delta + \xi\zeta\right] e_3 \tag{3.30}$$

$$R(e_1,e_2)e_3 = \left[-e_1(\xi) + \delta\alpha + e_2(\beta) + \eta\gamma - \xi\gamma - \xi\mu - \alpha\beta - \nu\mu\right] e_1 \tag{3.31}$$

$$+ \left[e_1(\delta) - \xi\alpha + e_2(\eta) - \beta\gamma - \gamma\delta - \alpha\eta - \eta\zeta - \xi\zeta\right] e_2 \tag{3.32}$$

$$R(e_1,e_3)e_2 = \left[-e_1(\nu) + \zeta\beta + e_3(\alpha) + \eta\mu + \eta\gamma - \beta\alpha + \nu\gamma + \eta\nu\right] e_1 \tag{3.33}$$

$$+ \left[-e_1(\zeta) - \nu\beta - e_3(\eta) - \alpha\mu + \eta\delta + \beta\eta + \nu\delta + \mu\zeta\right] e_3 \tag{3.34}$$

$$R(e_1,e_3)e_3 = \left[e_1(\mu) - \zeta\alpha + e_3(\beta) - \nu\eta - \eta\xi - \beta^2 - \nu\xi - \mu^2\right] e_1 \tag{3.35}$$

$$+ \left[e_1(\zeta) + \mu\alpha + e_3(\eta) + \beta\nu - \eta\delta - \beta\eta - \nu\delta - \mu\zeta\right] e_2 \tag{3.36}$$

Now, assuming finite volume, note that all the terms involving an e_1 derivative are zero since $e_1(f) = 0$ for all f constant along nullity geodesics. All terms involving e_2 or e_3 derivatives are zero since the Christoffel symbol they differentiate is zero. All other terms involve a Christoffel symbol which has been shown to be zero, except the $\zeta\xi$ terms, which were also shown to be zero. Hence, $R(e_1,\cdot)\cdot$ is identically zero, which is a contradiction.

This shows that the splitting tensor C is identically zero on V. So by Proposition 13, we have shown Theorem 11. □

Note that the hypothesis that M is 4-dimensional is used only to get the vector field T. In the case of conullity 2 n-manifolds, T was constructed in [FZ16] for any $n > 2$ by noting that C_T is zero if self-adjoint and therefore the image of $T \mapsto C_T$ is a one-dimensional subspace of 2×2 matrices. Hence T may be taken to be a vector field perpendicular to the kernel of $T \mapsto C_T$. Such a strategy fails for conullity 3 manifolds, since the space of self-adjoint matrices is only 6 dimensional for 3×3 matrices.

3.2. Nonnegative Curvature

Of Theorem 12. The proof is very similar to the conullity 2 result, but needs more work in certain areas. The assumption that $\sec \geq 0$ gives that M has a compact, totally geodesic soul S. As before, we first show that projections of nullity vectors are also nullity vectors.

Lemma 14. *If $T \in \ker R_p$ at a point $p \in S$, then the orthogonal projections $T^S \in T_pS$ and $T^N \in T_pS^\perp$ are also in $\ker R_p$.*

Proof. This is similar to the proof in the conullity 2 case. We write X^S and X^N for the orthogonal projections onto T_pS and $T_pS^\perp$ for any X. First observe that since $T = T^S + T^N \in \ker R_p$, that

$$\left\langle R(T^N, X)Y, Z\right\rangle = -\left\langle R(T^S, X)Y, Z\right\rangle \tag{3.37}$$

for any X, Y, Z.

Suppose for contradiction that T^N is not in $\ker R_p$. We may rescale T to make T^N unit length for simplicity. Equation 3.37 shows that it suffices to consider T^N to also show this result for T^S. We choose vectors U, V so that $\{T^N, U, V\}$ are orthonormal, U, V are each in either T_pS or $T_pS^\perp$ and so that they are not in $\ker R_p$. In particular, to see that $R(T^N, \cdot)\cdot = 0$, it suffices to see that

$$\left\langle R(T^N, X)Y, Z\right\rangle = 0 \tag{3.38}$$

for all $X, Y, Z \in \{T^N, U, V\}$. We now proceed through the possibilities for X, Y, Z.

First observe that we can assume $X \neq T^N$ and $Y \neq Z$ by the symmetries of R. Hence one of Y or Z are either X or T^N, and we assume that it is Y. Then either Z is perpendicular to X and T^N or we have $Y = X$, $Z = T^N$. This gives us three cases to examine:

$$(a) = \left\langle R(T^N, X)T^N, X \right\rangle \tag{3.39}$$

$$(b) = \left\langle R(T^N, X)X, Z \right\rangle \tag{3.40}$$

$$(c) = \left\langle R(T^N, X)T^N, Z \right\rangle \tag{3.41}$$

where $Z \perp X, T^N$.

Case (a) is just $-\sec(T^N, X)$. Either $X \in T_pS$ or $X \in T_pS^\perp$. We denote by superscripts S and N the two cases. Observe that $\sec(T^N, X^S)$ is zero since it is the curvature of a flat strip from Perelman's proof of the Soul Conjecture. Next observe that Equation 3.37 shows, for the case of $X^N \in T_pS^\perp$, that $\sec(T^N, X^N) = \sec(T^S, X^N)$ which is again the curvature of a flat strip and so is zero. Hence $(a) = 0$.

For case (b), we similarly consider the two possibilities of $X^S \in T_pS$ or $X^N \in T_pS^\perp$. The first gives $R(T^N, X^S)X^S$ which must be a vector in the span of T^N and X^S since the flat strips are also totally geodesic, and hence the inner product with Z is zero. For the other case, we apply Equation 3.37 again and see that

$$\left\langle R(T^N, X^N)X^N, Z \right\rangle = -\left\langle R(T^S, X^N)X^N, Z \right\rangle = 0 \tag{3.42}$$

for the same reason.

For case (c) we do the same as in (b). □

This gives, as in conullity 2, the result that there is an orthonormal basis $B = \{e_1, e_2, e_3, T\}$ of T_pM, for each $p \in S$, with T in $\ker R$ and e_i in $\ker R^\perp$ and so that each e_i and T is in either T_pS or in $T_pS^\perp$. However, our assumptions are not enough to conclude that either all of e_1, e_2, e_3 are in T_pS or that all of them are in $T_pS^\perp$. So instead we have the cases that either zero, one, two, or three of e_1, e_2, e_3 are in T_pS, and by continuity of $\ker R$, whichever of these holds at one point on S must hold for all points of S.

First consider the cases where either none or exactly one of the e_i lie in T_pS, which then implies that S is flat. By Lemma 10, either $\tilde{M}$ splits with a Euclidean factor, or S is a point.

Next, consider the case where all three of the e_i lie in T_pS. If T lies in T_pS as well, S is four dimensional, so $S = M$. Then M is compact and so splits by Theorem 11. If instead, T lies in $T_pS^{\perp}$, then S is a codimension 1 soul and so M splits isometrically as $S \times \mathbb{R}$ [CG72].

Finally, consider the case where $e_1, e_2 \in T_pS$ but $e_3 \in T_pS^{\perp}$. If $T \in T_pS$, then S is codimension 1 and again M splits isometrically as $M = S \times \mathbb{R}$. So assume that $T \in T_pS^{\perp}$. For $i, j \in \{1, 2\}$, observe that, since S is totally geodesic,

$$\langle \nabla_{e_i} e_3, e_j \rangle = - \langle e_3, \nabla_{e_i} e_j \rangle = 0, \tag{3.43}$$

and also

$$\langle \nabla_{e_i} e_3, e_3 \rangle = \frac{1}{2} e_i(\langle e_3, e_3 \rangle) = 0. \tag{3.44}$$

Since e_i, T span a flat totally geodesic strip,

$$\langle \nabla_{e_i} e_3, T \rangle = - \langle e_3, \nabla_{e_i} T \rangle = 0 \tag{3.45}$$

and so we get $\nabla_{e_i} e_3 = 0$. Similarly, $\nabla_T e_3 = 0$. These show that e_3 and T are normal parallel vector fields, though maybe defined only locally. Suppose that M is simply connected. Then e_3 and T would be globally-defined parallel normal vector fields on S. And hence M is isometric to the space of all souls and hence splits isometrically as $M = S \times \mathbb{R}^2$ [Yim90, Mar96, Str88]. This completes the proof for the case that M is simply connected.

For this last case with M not simply connected, we then know that the universal cover $\widetilde{M}$ either splits isometrically or $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^4$. In the first case, we are done, so we assume that $\widetilde{M} \approx \mathbb{R}^4$. In the current case, M itself has $e_1, e_2 \in T_pS$ and $e_3, T \in T_pS^{\perp}$. So, M has a 2 dimensional soul S. Either S is flat or there is at least one point on S where

$\sec(e_1, e_2) > 0$. In the first case, Lemma 10 gives that $\widetilde{M}$ must split.

So suppose that S has a point where $\sec(e_1, e_2) > 0$. Then by Gauss-Bonnet, $\widetilde{S}$ must be a sphere. Since M is diffeomorphic to the normal bundle $\nu(S)$, then $\widetilde{M}$ is diffeomorphic to the universal cover of $\nu(S)$, which is the pullback bundle $\pi^*(\nu(S))$ by $\pi : \widetilde{M} \to M$. This pullback bundle is a vector bundle over $\widetilde{S}$, a sphere. This contradicts the fact that $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^4$. □

As in the conullity 2 result, there is no known example of the case where M is diffeomorphic to $\mathbb{R}^4$ and does not split a Euclidean factor.

3.3. Example

The following example is a modification of the conullity 2 example from Sekigawa given in (1.4). Let M^4 be $\mathbb{R}^4$ with coordinates x, u, v, w and define the metric on M by

$$g = (p(x,u,w)dx)^2 + (du - (v+w)dx)^2 + (dv + (u+w)dx)^2 + (dw - (v-u)dx)^2 \quad (3.46)$$

with $p > 0$. Then (M, g) has conullity 3. In particular, define

$$T := \tfrac{\partial}{\partial v}, \quad (3.47)$$

$$e_1 := \frac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial u} + \tfrac{\partial}{\partial w}\right), \quad (3.48)$$

$$e_2 := \frac{1}{p(x,u,w)}\left(\tfrac{\partial}{\partial x} + (v+w)\tfrac{\partial}{\partial u} - (u+w)\tfrac{\partial}{\partial v} + (v-u)\tfrac{\partial}{\partial x}\right), \quad (3.49)$$

$$e_3 := \frac{1}{\sqrt{2}}\left(\tfrac{\partial}{\partial u} - \tfrac{\partial}{\partial w}\right). \quad (3.50)$$

This gives an orthonormal basis with T the nullity direction and the splitting tensor C_T acting on $\{e_1, e_2, e_3\}$ is

$$C = \begin{pmatrix} 0 & \frac{\sqrt{2}}{p} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.51)$$

The $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ vectors integrate to geodesics, as do the e_1, e_3 vectors. The hyperplanes given by span $\{T, e_1, e_3\} = \text{span}\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}\right\}$ integrate to flat, totally geodesic submanifolds. The scalar curvature is

$$\text{Scal} = \frac{-2}{p}\left(\frac{\partial^2 p}{\partial w^2} + \frac{\partial^2 p}{\partial u^2}\right). \tag{3.52}$$

The curvature tensor R is zero except when involving $\frac{\partial}{\partial x}$. And $R\left(\frac{\partial}{\partial x}, \cdot\right)\frac{\partial}{\partial x} : T_pM \to T_pM$ is given by the matrix

$$-\begin{pmatrix} \frac{\partial^2 p}{\partial u^2} & \frac{\partial^2 p}{\partial u \partial w} \\ \frac{\partial^2 p}{\partial u \partial w} & \frac{\partial^2 p}{\partial w^2} \end{pmatrix} \tag{3.53}$$

in in the basis of $\frac{\partial}{\partial u}, \frac{\partial}{\partial w}$ (with all terms involving $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial v}$ being zero). Note that the plane spanned by $\{e_1, e_3\}$ is flat so that (M, g) does not satisfy the assumptions of the result in [Ros67] for any p.

Note that we cannot have Scal > 0 everywhere on a complete manifold of this form, since then, fixing a particular x and considering p as a function just of u and v, then $\Delta p < 0$ everywhere and so p must have a zero which makes the metric singular. Hence we cannot have sec ≥ 0 with conullity exactly 3 either, as is guaranteed by Theorem 12.

CHAPTER 4 : Constant Ricci Eigenvalues $(\lambda, \lambda, 0)$

In this chapter, we study the Riemannian 3-manifolds (M^3, g) whose Ricci tensor has constant eigenvalues $(\lambda, \lambda, 0)$, which we will see reduces to the case where $\lambda = -1$.

4.1. Preliminaries

We begin by observing that if M has Ricci eigenvalues $(\lambda, \lambda, 0)$, then M is conullity 2 (even without λ constant). In particular, if T is the eigenvector for the 0 eigenvalue of the Ricci tensor, then $T \in \ker R$ and if $\lambda \neq 0$, then clearly $R \neq 0$ so $\dim \ker R = 1$. We define T to be the (unique up to sign) unit eigenvector field on M, passing to a double cover if necessary to get a global field. We may use the conventions of 1.1 for conullity 2 manifolds. Define C to be the splitting tensor of T.

Along a nullity geodesic,

$$\mathrm{Scal}' = -2 \operatorname{tr} C. \tag{4.1}$$

To see this, use second Bianchi identity and the fact that $\mathrm{Scal} = \langle R(X,Y)Y, X\rangle$ to imply that

$$\mathrm{Scal}' = \langle (\nabla_T R)(X,Y)Y, X\rangle = \langle R(Y, \nabla_X T)Y, X\rangle + \langle R(\nabla_Y T, X)Y, X\rangle \tag{4.2}$$

$$= -\,\mathrm{Scal} \cdot \langle C(X), X\rangle - \mathrm{Scal} \cdot \langle C(Y), Y\rangle\,. \tag{4.3}$$

Since M is curvature homogeneous, we must then have that $\operatorname{tr} C$ is zero. Note that (1.9) implies that $(\operatorname{tr} C)' = \operatorname{tr}(C^2) = (\operatorname{tr} C)^2 - 2\det C$ along a nullity geodesic. Hence $\det C = 0$ as well. Since the only real eigenvalues of C are zero, we are in the case where C is nilpotent.

We define M_0 to be the subset of M where $C = 0$ and M_C to be the complement, where $C \neq 0$.

Hence we can construct a local orthonormal basis e_1, e_2, T on M_C defined by $T \in \ker R$ and $e_1 \in \ker C$. We will be interested in the case where both M_0 and M_C are non-empty, so

e_1 and e_2 may not be defined globally. Since $C' = 0$, e_1 and e_2 are parallel along nullity geodesics. There exists a smooth function $a \neq 0$ on M_C so that $C(e_2) = ae_1$. Hence

$$\nabla_T e_1 = \nabla_T e_2 = \nabla_T T = 0, \quad \nabla_{e_1} T = 0, \quad \nabla_{e_2} T = -ae_1 \tag{4.4}$$

$$\nabla_{e_1} e_1 = \alpha e_2, \quad \nabla_{e_2} e_2 = \beta e_1, \quad \nabla_{e_1} e_2 = -\alpha e_1, \quad \nabla_{e_2} e_1 = aT - \beta e_2 \tag{4.5}$$

for some smooth functions α, β on M_C. Thus for the curvature tensor we have

$$R(e_2, e_1)e_1 = (e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2)e_2 + (a\beta - e_1(a))T \tag{4.6}$$

$$R(e_1, e_2)e_2 = (e_1(\beta) + e_2(\alpha) - \alpha^2 - \beta^2)e_2 + \alpha a T. \tag{4.7}$$

Since $T \in \ker R$, this implies that

$$\alpha = 0, \quad \mathrm{Scal}_M = e_1(\beta) - \beta^2, \quad \text{and} \quad e_1(a) = a\beta. \tag{4.8}$$

Lemma 15. *Let M^3 be complete with constant Ricci eigenvalues $(\lambda, \lambda, 0)$ with $\lambda \neq 0$ Then*

(a) $\lambda = -1$, up to scaling,

(b) integral curves of e_1 and T starting at points in M_C are complete geodesics contained in M_C,

(c) the distribution in M_C spanned by e_1 and T is completely integrable with flat, totally geodesic leaves contained in M_C,

(d) at all points in M_C, we have $|\beta| \leq 1$.

Proof. Take $p \in M_C$. Then e_1 is well-defined in a neighborhood of p in M_C. Since $\nabla_{e_1} e_1 = 0$, the integral curve of e_1 is locally a geodesic η which is defined so long as $C \neq 0$. The distribution spanned by e_1, T is integrable by (4.4), totally geodesic since $\alpha = 0$, and flat since it contains T. We need to show that the complete geodesics η lie in M_C.

Writing a dot to indicate e_1 derivatives, we get

$$\left(\frac{1}{a}\right)^{\cdot\cdot} = -\left(\frac{\dot{a}}{a^2}\right)^{\cdot} = -\left(\frac{\beta}{a}\right)^{\cdot} = -\frac{(\mathrm{Scal}+\beta^2)}{a} + \frac{\beta^2}{a}. \tag{4.9}$$

Hence

$$\left(\frac{1}{a}\right)^{\cdot\cdot} + \frac{1}{a}\,\mathrm{Scal} = 0 \tag{4.10}$$

and so $\frac{1}{a}$ satisfies the Jacobi equation. This equation holds only in M_C. We must then show that a cannot go to zero along η.

If Scal $= 2\lambda$ is positive, then $\frac{1}{a}$ has solutions of the form $\frac{1}{a} = A_0 \cos t + A_1 \sin t$ which is bounded and hence a never goes to zero. Therefore η remains in M_C. But then there is a zero of $\frac{1}{a}$ in finite time which implies that a diverges. This is a contradiction since C is well-defined on all of M. Hence we may assume that $\lambda = -1$ by rescaling the metric.

Thus the solutions are of the form $\frac{1}{a} = A_0 \cosh(t) + A_1 \sinh(t)$. Hence $a \to 0$ only as $t \to \pm\infty$ and therefore C remains non-zero along η for all time. Since (1.9) implies that C is constant along nullity geodesics as well, C cannot go to zero on any leaf of the span of $\{e_1, T\}$ and hence the leaf is complete.

Since $\beta = e_1(a)/a = -ae_1(1/a)$,

$$\beta(t) = -\frac{A_0 \sinh(t) + A_1 \cosh(t)}{A_0 \cosh(t) + A_1 \sinh(t)} = -\frac{\tanh(t) - \beta(0)}{1 - \beta(0)\tanh(t)}.$$

This implies that $|\beta| \leq 1$ since otherwise β has a singularity in finite time along the complete geodesic η. □

4.2. Local Metric

The form of metrics with Ricci eigenvalues $(-1,-1,0)$ is well-known locally at points where $C \neq 0$ [Sek75, Sza85, KTV92, KTV90]. This is usually stated in the following form.

$$(A(x)\cosh u + B(x)\sinh u)^2 dx^2 + (du - v\ dx)^2 + (dv + u\ dx)^2 \tag{4.11}$$

We will give a geometric proof of this result but with a slightly different parametrization (differing only in the x coordinate) which is better suited to handling the case when $C = 0$. This form will be

$$g = (\cosh u - h(x)\sinh u)^2 dx^2 + (du - f(x)v\ dx)^2 + (dv + f(x)u\ dx)^2 \tag{4.12}$$

Moreover, we will show that this form holds in a "global" sense: that such a coordinate chart covers an entire connected component of M_C when M is simply connected and complete. Observe that metrics of this form are just the product metric $\mathbb{H}^2 \times \mathbb{R}$ when $f(x) = 0$ and hence do support $C = 0$. (Taking $f(x) = 0$ and $h(x) = 0$ gives the u, v coordinates a standard parametrization of $\mathbb{H}^2$. However, any values for $h(x)$ will still locally give $\mathbb{H}^2$ and those that satisfy $|h(x)| \leq 1$ will be complete as well by the following lemma.)

We begin with two technical lemmas. The first considers the completeness of metrics which have the form of (4.12).

Lemma 16. *Suppose g is a metric on $V = (a_1, a_2) \times \mathbb{R}^2$, with coordinates $x \in (a_1, a_2)$ and $u, v \in \mathbb{R}^2$, of the form (4.12) with $h, f : (a_1, a_2) \to \mathbb{R}$. Then*

(a) g has Ricci eigenvalues $(-1,-1,0)$,

(b) V is complete if and only if $(a_1, a_2) = (-\infty, \infty)$ and $|h(x)| \leq 1$ for all x, and

(c) g is locally irreducible if and only if $f^{-1}(0)$ contains no open subsets.

Proof. That the Ricci eigenvalues are $(-1,-1,0)$ follows by direct computation.

We consider completeness. Note that $|h(x)| \leq 1$ when complete is a consequence of Lemma 15 by computing that $h(x) = \beta$ at the point $(x, 0, 0)$.

Let

$$\bar{x} = \int_0^x f(X) dX$$

a function of x. We perform a change-of-coordinates by

$$(x, y, z) = (x, u \cos \bar{x} - v \sin \bar{x}, u \sin \bar{x} + v \cos \bar{x}).$$

This performs a rotation in each u-v plane by an amount that depends on x. In these coordinates, g is

$$g = p(x, y, z)^2 dx^2 + dy^2 + dz^2$$

where $p(x, y, z) = \cosh(u) - h(x) \sinh(u)$ with $u(x, y, z) = y \cos \bar{x} + z \sin \bar{x}$. This is a more explicit form of the metric in Theorem 2.5 of [Sza85].

We will prove the contrapositive statement; g is not complete if and only if the interval (a_1, a_2) has a_1 or a_2 finite. Suppose that g is not complete. Then there is a path γ of finite length which has no limit in V. Let $\gamma(t) = (x(t), y(t), z(t))$. Then $\int |y'(t)| \, dy$ and $\int |z'(t)| \, dt$ are both lower bounds for the length of γ. Hence $y(t)$ and $z(t)$ are bounded. In particular, this gives that $|u| \leq R$ for some $R \in \mathbb{R}$.

Moreover, $\int |\cosh(u) - h(x) \sinh(u)| \, |x'(t)| \, dt$ is also a lower bound for the length of γ and hence is finite. Then

$$|\cosh(u) - h(x) \sinh(u)| \geq 2e^{-R}$$

holds since $|h(x)| \leq 1$. Thus $\int |x'(t)| \, dt$ is finite, and so either a_1 or a_2 must be finite.

For the other direction, either a_1 or a_2 is finite. Without loss of generality, we will assume that a_2 is finite. Then consider the path $\gamma(x) = (x, 0, 0)$ for $x \in [m, a_2)$. This path has length

$$\int_m^{a_2} dx$$

which is finite but has no limit in V. Hence V is not complete. $\square$

The following technical lemma will be used in Proposition 18 and Theorem 31.

Lemma 17. *Suppose that a manifold M^n (without boundary) is decomposed as $M = \bar{U} \cup \bar{V}$ where $\bar{U}$ and $\bar{V}$ are the closures of two open, disjoint subsets of M with $\bar{U} \cap \bar{V}$ a closed $n-1$ dimensional sub-manifold. If M is simply connected and each boundary component of $\bar{U}$ is simply connected, then U and V are also simply connected.*

Proof. Note that it suffices to prove that U is simply connected since the boundary components of U are also the boundary components of V. Suppose that $\tilde{U}$ is a covering of $\bar{U}$. We will use this to construct a covering of M which shows that $\tilde{U}$ must be a trivial cover.

Let $\{B_i\}$ be the set of boundary components of U. Since each B_i is simply connected, there are lifts of $B_i \hookrightarrow \bar{U}$ to $\tilde{U}$ as well as lifts of γ_i to $\tilde{U}$. For each i, pick γ_i to be a path starting at B_0 and ending at B_i.

Fix a lift $\tilde{B}_0$ of B_0, which fixes for each i a unique lift $\tilde{\gamma}_i$ that starts in $\tilde{B}_0$, for each i. These then determine a unique lift $\tilde{B}_i$ of each B_i such that $\tilde{\gamma}_i$ ends at $\tilde{B}_i$. Then glue a copy of V onto $\tilde{U}$ by identifying the boundary component B_i of V with the boundary component $\tilde{B}_i$ of $\tilde{U}$.

Doing this for all possible lifts $\tilde{B}_0$ gives a covering space of M. Hence $\tilde{U}$ is a trivial cover. $\square$

Next we find the metric in the parts of M where $C \neq 0$.

Proposition 18. *Suppose that M is a complete, simply connected Riemannian 3-manifold with Ricci eigenvalues $(-1, -1, 0)$. Then any connected component of M_C has coordinates*

$(x,u,v) \in (a_1,a_2)\times\mathbb{R}^2$ *(with a_i possibly $\pm\infty$) with metric of the form*

$$g = (\cosh(u) - h(x)\sinh(u))^2 dx^2 + (du - vf(x)\,dx)^2 + (dv + uf(x)\,dx)^2 \tag{4.13}$$

for some smooth functions $h, f : (a_1,a_2) \to \mathbb{R}$ with $f(x) \neq 0$ and $|h(x)| \leq 1$. The boundaries of this component are complete, flat, totally geodesic planes, one for each a_i that is finite.

Proof. Let M_C° be a connected component of M_C. Fix any $\gamma : (a_1,a_2) \to M$, a maximal integral curve of the e_2 vector field on M_C°. Let N be the manifold defined by one coordinate chart with $(x,u,v) \in (a_1,a_2)\times\mathbb{R}^2$ and metric of the form in (4.13) where $f(x) = a(\gamma(x))$ and $h(x) = -\beta(\gamma(x))$, By Lemma 15 this implies that $|h| \leq 1$.

This manifold is simply connected but may not be complete. Define $\phi : N \to M$ by

$$\phi(x,u,v) = \exp_{\gamma(x)}(ue_1 + vT).$$

We will show that this map is in fact an isometry onto M_C°.

We first show that this is a local isometry. Note that $\frac{\partial\phi}{\partial u} = e_1$ and $\frac{\partial\phi}{\partial v} = T$. We then must compute $\frac{\partial\phi}{\partial x}$. Fix (x_0,u_0,v_0). Consider the family of geodesics $\alpha_s(t) = \phi(x_0+s, tu_0, tv_0)$. We construct the Jacobi field $J(t)$ corresponding to α along the geodesic $\alpha_0(t)$. Choose $J(0) = e_2 = \gamma'(0)$ and $J'(0) = \nabla_{e_2}(u_0e_1 + v_0T) = u_0(aT - \beta e_2) - v_0ae_1$. Computing using (4.4)–(4.8) gives that

$$J(t) = -v_0a(\gamma(x_0))te_1 + [\cosh(u_0t) - \beta(\gamma(x_0))\sinh(u_0t)]\,e_2 + u_0a(\gamma(x_0))tT.$$

By the definition of $f(x) = a(\gamma(x))$ and $h(x) = \beta(\gamma(x))$, we see that

$$\left.\frac{\partial\phi}{\partial x}\right|_{(x_0,u_0,v_0)} = J(1) = (\cosh u_0 - h(x_0)\sinh u_0)\,e_2 - v_0f(x)e_1 + u_0f(x)T.$$

Now it is easy to check that the inner products of $\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial v}$ in M are the same as the inner products of $\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ in N. So ϕ is a local isometry.

Consider the image $\phi(N) \subset M$. By Lemma 16, the boundary of $\phi(N)$ has at most two connected components, each of which is the image of a flat, totally-geodesic plane, i.e. the limits of the u, v planes going to either a_1 or a_2 if they are non-infinite. By Lemma 20, the boundary components of $\phi(N)$ are flat, totally-geodesic planes in M. Hence Lemma 17 implies that $\phi(N)$ is simply connected.

We now want to argue that ϕ is a covering map of $\phi(N)$ and hence an isometry onto $\phi(N)$. We do this by showing that ϕ has the path lifting property. Suppose that there is a path $\mu : [0,1] \to \phi(N)$. Then there is a lower bound $a > \delta > 0$ on μ. Since ϕ is a local isometry, we can lift μ in a neighborhood of a point to get $\tilde{\mu} : [0, \epsilon) \to N$. Since a is an isometry invariant up to sign and a on μ is bounded away from zero, the path $\tilde{\mu}$ must have a bounded away from zero as well. Hence $\lim_{t\to\epsilon} \tilde{\mu}(t)$ is a point in the interior of N. Since ϕ is a local isometry, we can then extend the lifting of μ past $t = \epsilon$. Hence, ϕ has the path lifting property and is a covering map. Therefore ϕ is a covering map and since its image is simply connected, ϕ is an isometry onto its image.

It remains to be shown that $M_C^\circ = \phi(N)$. By Lemma 16, the boundary components of N, and hence of $\phi(N)$ as well, are planes corresponding to any finite endpoint of γ. Since $C \to 0$ on any endpoints of γ, $C \to 0$ on any boundary planes of N. Note that M_C° contains $\phi(N)$ and that $\phi(N)$ is open. We claim that $\phi(N)$ is also closed. If $x \in M_C^\circ$ is not in $\phi(N)$ then take some neighborhood U of x contained in M_C°. Then $C \neq 0$ on U and we claim that $U \cap \phi(N)$ is empty. If $U \cap \phi(N)$ is non-empty, then there is a path from a point $y \in \phi(N)$ to $x \in M_C^\circ \setminus \phi(N)$ which lies in M_C°. This path cannot cross the boundary planes of $\phi(N)$ since those planes have $C = 0$. Hence, by the completeness argument in Lemma 16, the path must lie in $\phi(N)$. This is a contradiction. So M_C° is connected and $\phi(N)$ is an open and closed subset, hence $M_C^\circ = \phi(N)$. This shows that M_C° has the desired coordinate chart and its metric is of the form (4.13). □

Remark 19. Now, we state important properties of this metric which we will use throughout this paper. If g has the form of (4.13), with an $f(x)$ arbitrary smooth function and $|h(x)| \leq 1$, then

(a) Ricci has eigenvalues $(-1,-1,0)$,

(b) $T = \frac{\partial}{\partial v}$ is in $\ker R$ and hence is the eigenvector of Ricci with eigenvalue 0,

(c) $e_1 = \frac{\partial}{\partial u}$ is in the kernel of C,

(d) $e_2 = (\cosh(u) - h(x)\sinh(u))^{-1}\left(\frac{\partial}{\partial x} + vf(x)\frac{\partial}{\partial u} - uf(x)\frac{\partial}{\partial v}\right)$ is such that $\{e_1, e_2, T\}$ is an orthonormal basis which satisfies the properties calculated in Section 4.1,

(e) the subsets where x is constant are complete, flat, totally geodesic planes spanned by $\{e_1, T\}$ and these planes are the leaves of the foliation $\mathcal{F}$,

(f) every point (x, u, v) such that $f(x) \neq 0$ has an irreducible neighborhood,

(g) $a = f(x)(\cosh(u) - h(x)\sinh(u))^{-1}$, so $C = 0$ at (x, u, v) if and only if $f(x) = 0$,

(h) $\beta = (h(x)\cosh u - \sinh(u))(\cosh u - h(x)\sinh(u))^{-1}$,

(i) in particular, $f(x) = a((x, 0, 0))$ and $h(x) = \beta((x, 0, 0))$, and

(j) $f(x) = 0$ on an interval (a, b) if and only if $(a, b) \times \mathbb{R}^2$ is locally isometric to $\mathbb{H}^2 \times \mathbb{R}$.

4.3. Foliation by Flat Planes

We now discuss the properties of the foliation on M_C from 15. This foliation is the defining geometric property of these manifolds. We will see that it extends to a (not necessarily smooth) foliation on the closure $\overline{M_C}$, that there are curves orthogonal to the foliation everywhere, and that the connected components of $\overline{M_C}$ are plane bundles over these curves.

Lemma 20. *Let M^3 be a complete Riemannian manifold with Ricci eigenvalues $(-1,-1,0)$. Then the closure $\overline{M_C}$ of M_C is foliated by complete, flat, totally geodesic surfaces whose tangent planes form a continuous distribution which is smooth in M_C. We call this foliation $\mathcal{F}$ through each $p \in \overline{M_C}$.*

Proof. By Lemma 15, through every point $x \in M_C$ there is a complete, flat, totally geodesic leaf P_x. Every convergent sequence of points $x_k \to x$ has a subsequence such that the leaves $P_{x_{k_j}}$ converge to a surface P at x. So it suffices to show that if $x_k \to x$ and $y_k \to x$ with $P_{x_k} \to P$ and $P_{y_k} \to Q$, that $P = Q$. Assume that P is not Q.

There is a smooth function $F : M \to \mathbb{R}$ in a neighborhood U of x such that $F = 0$ on Q, $F > 0$ on one connected component of $U \setminus Q$ and $F < 0$ on the other. Then there exists $p \in P$ such that $F(p) > 0$ and another point $p' \in P$ such that $F(p') < 0$. Since F is smooth, for large enough k, there must exist points in P_{x_k} in the same connected component of $M \setminus Q$ as p and hence have $F > 0$. Similarly, some points must have $F < 0$. Hence, for large enough k, some point of P_k has $F = 0$ and $P_k \cap Q$ is non-empty. This is a contradiction since $P_k \subset M_C$, but Q has $C = 0$ at every point since $C \to 0$ on P_{y_k}.

Therefore the foliation is well-defined and continuous. □

The following proposition is stated in more generality than we need since it is of some interest on its own. This result is presumably already known but we give the proof for completeness.

Proposition 21. *Let M^n be a complete Riemannian manifold with $-1 \leq \sec \leq 0$ and let V be any open subset of M. Suppose that $\mathcal{F}$ is a foliation of V whose leaves are complete, totally geodesic hypersurfaces. Then $\mathcal{F}$ is Lipschitz in the sense that if γ is any path parametrized by arc-length and N is a unit normal vector field to $\mathcal{F}$, then $f(t) = \langle N, V(t) \rangle$ is Lipschitz for any parallel vector field $V(t)$ along γ.*

Proof. It suffices to show this in the case where M is simply connected since the result is local. We first begin by considering two special cases of curves and then showing it in general by approximating arbitrary curves with these two simpler cases. The first case is when γ is a geodesic in a leaf of $\mathcal{F}$. Then $f(t)$ is constant and so is trivially Lipschitz.

For the second case, we will assume that γ is a geodesic such that $\gamma'(0)$ is perpendicular

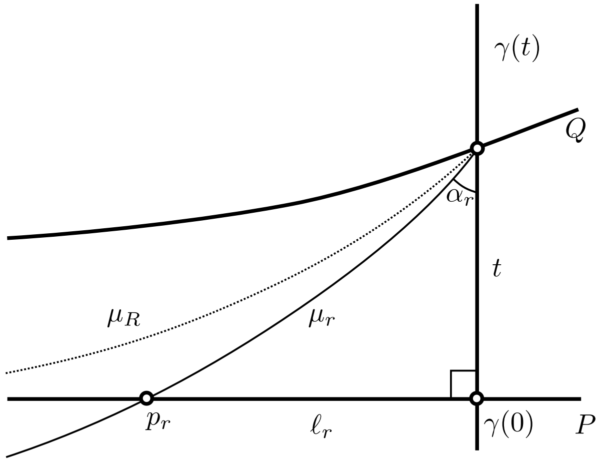

Figure 1: Diagram of the triangles in the case where γ is a geodesic which starts orthogonal to the foliation.

to $\mathcal{F}$. Instead of showing that $f(t)$ is Lipschitz we will only show $|f(0) - f(t)| \leq t$ for all t. Fix $t > 0$. Let P be the leaf of $\mathcal{F}$ at $\gamma(0)$ and Q the leaf at $\gamma(t)$. Let $X \in T_{\gamma(t)}M$ be any unit vector in $\mathcal{F}$. Consider the geodesics $\mu_r(s) = \exp_{\gamma(t)}(s(rX - (1-r)\gamma'(t)))$. Note that μ_0 intersects P at $\gamma(0)$ and μ_1 lies in Q and hence does not intersect P. Moreover, if μ_r intersects P, then it must do so transversely since P is totally geodesic but cannot intersect Q. Hence the set of $r \in [0,1]$ that intersect P is open. Therefore there is a minimal $R \in (0,1)$ such μ_R does not intersect P.

For all $0 \leq r < R$, μ_r intersects P at some point p_r, which is unique since geodesics are unique in M. Let ℓ_r be the distance from $\gamma(0)$ to p_r. Then there is a geodesic triangle with vertices $\gamma(0)$, $\gamma(t)$, and p_r. The angle at $\gamma(0)$ is $\pi/2$, the length of the side opposite $\gamma(t)$ is ℓ_r and the side opposite p_r has length t. Let α_r be the angle at $\gamma(t)$. See Figure 1.

Now construct a comparison triangle in $\mathbb{H}^2$ with one angle of $\pi/2$ and the two adjacent sides of lengths ℓ_r and t. Let $\bar{\alpha}_r$ be the angle in the comparison triangle opposite the side of length ℓ_r. Then since M has sec ≥ -1, $\alpha_r \geq \bar{\alpha}_r$. Since μ_R does not intersect P, we must have that $\ell_r \to \infty$ as $r \to R$. We now examine what $\bar{\alpha}_r$ does in the comparison triangle as $\ell_r \to \infty$. This angle approaches the so-called angle of parallelism, $\Pi(t)$. This angle is defined such that any ideal triangle in $\mathbb{H}^2$ with angles 0, $\pi/2$, and $\Pi(t)$ has t as the side length of its only finite edge. (The angle 0 occurs at the ideal point.) It satisfies $\cos\Pi(t) = \tanh t$ and, equivalently, $\sin\Pi(t) = \operatorname{sech} t$. In our case we get that $\bar{\alpha}_r \to \Pi(t)$ as $r \to R$. Hence $\alpha_R \geq \Pi(t)$.

Therefore the angle from Q to $-\gamma'(t)$ must be greater than $\Pi(t)$ as well. Hence the normal vector N of Q has angle at most $\pi/2 - \Pi(t)$ from $\gamma'(t)$. This implies that

$$\left|\left\langle\gamma'(0), N\right\rangle - \left\langle\gamma'(t), N\right\rangle\right| \leq |1 - \cos(\pi/2 - \Pi(t))| = 1-\sin(\Pi(t)) = 1-\operatorname{sech} t \leq t^2/2. \quad (4.14)$$

This further implies that $|f(t)| \leq t$ for any unit vector field V parallel along γ. The case with $V = \gamma'$ is done by (4.14) and for $V \perp \gamma'$ we can see that since $\langle\gamma'(t), N\rangle > 1 - t^2/2$ we must have that $\langle V, N\rangle \leq t$ since $\langle N, V\rangle^2 + \langle N, \gamma'(t)\rangle^2 \leq 1$.

Now we consider a general path $\gamma : [0, T] \to M$ parametrized by arc length which we approximate with segments of the form above. For any $\delta > 0$, we may pick points $0 = t_0 < t_1 \ldots < t_N = T$ such that $d(\gamma(t_{k+1}), \gamma(t_k)) < \delta$ for all k. For each k, let p_k be the point in the leaf P_k of $\mathcal{F}$ at $\gamma(t_k)$ such that p_k is closest to $\gamma(t_{k+1})$. Construct a piecewise geodesic path $\bar{\gamma}(t)$ by connecting by geodesics $\gamma(t_0)$ to p_0 to $\gamma(t_1)$ and in general connecting $\gamma(t_k)$ to p_k to $\gamma(t_{k+1})$. The length of the geodesic from p_k to $\gamma(t_{k+1})$ is at most the distance from $\gamma(t_k)$ to $\gamma(t_{k+1})$ so is at most δ. The distance from $\gamma(t_k)$ to p_k is at most 2δ since it is less than the distance from $\gamma(t_k)$ to $\gamma(t_{k+1})$ and then to p_k. Hence the length of this path is bounded above as $\ell(\bar{\gamma}) \leq 3\ell(\gamma)$ and every point on $\bar{\gamma}$ is at most δ from a point on γ. See Figure 2.

Note that the path from each p_k to $\gamma(t_{k+1})$ starts orthogonal to $\mathcal{F}$. This gives us a construction of arbitrarily good C^0 approximations to γ by geodesics that either lie in a leaf of $\mathcal{F}$ or who start perpendicular to $\mathcal{F}$. Let $\bar{V}$ be a parallel vector field along $\bar{\gamma}$ with $\bar{V} = N$ at $\bar{\gamma}(0)$ Since N is parallel along the segments from $\gamma(t_k)$ to p_k, we need only to consider the case segments which start orthogonal to $\mathcal{F}$ from p_k to $\gamma(t_{k+1})$. By our calculation above,

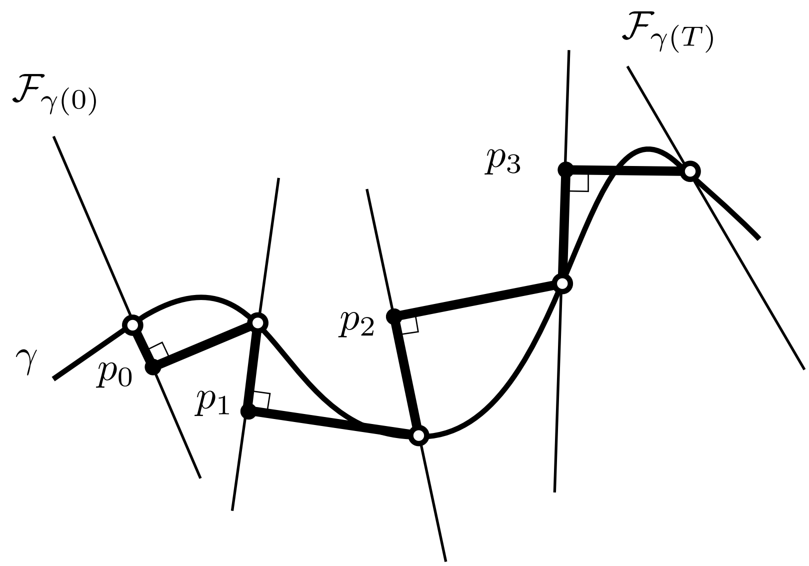

Figure 2: Approximation of an arbitrary path γ by geodesics that either lie in $\mathcal{F}$ or start perpendicular to $\mathcal{F}$. The thick path is $\bar{\gamma}$ approximating γ.

we get that

$$\left|\left\langle N, \bar{V}(0)\right\rangle - \left\langle N, \bar{V}(T)\right\rangle\right| \leq N\delta. \tag{4.15}$$

Now we want to conclude the same for the vector field V along γ with $V(0) = \bar{V}(0)$. Note that $\bar{\gamma}$ is an approximation of γ so the parallel transport of $V(0)$ along $\bar{\gamma}$ and along γ differ by the holonomy along the triangles with vertices $\gamma(t_k)$ to p_k to $\gamma(t_{k+1})$. Since the side lengths are all at most 2δ and we have sectional curve bounds, the Ambrose-Singer theorem says that the holonomy of around each triangle is on the order of δ^2 and hence the total holonomy goes to 0 as $\delta \to 0$. Hence $\bar{V}(T) \to V(T)$ as $\delta \to 0$ and since we may choose the points t_k such that $N\delta \to T$,

$$|\langle N, V(0)\rangle - \langle N, V(T)\rangle| \leq T. \tag{4.16}$$

This shows that $f(t)$ is Lipschitz since the parametrization of γ was arbitrary and hence we could have compared any two points on γ and not simply $\gamma(0)$ and $\gamma(T)$. □

Corollary 22. *Fix coordinates $(x_1, \ldots, x_n)$ on a neighborhood of M and let U be a relatively* *compact subset of that neighborhood. The vector field N normal to $\mathcal{F}$ is also Lipschitz in the sense that $\left\langle N, \frac{\partial}{\partial x_i} \right\rangle$ is Lipschitz as a function on M and hence there is a C^1 integral curve of N through any point of M.*

Proof. Let $(x_1, \ldots, x_n)$ be coordinates on some coordinate chart on M. We restrict to a relatively compact subset U of this chart. Let X_i denote $\frac{\partial}{\partial x_i}$. We first claim that there exists a universal constant C such that for any arc-length parametrized path γ and any unit vector V parallel along γ, that $\langle V, X_i \rangle$ is Lipschitz with constant C for any i. We first show this for geodesics γ. The space of all parallel vector fields along geodesics in $\bar{U}$ is $T^1\bar{U} \times U$, where $T^1\bar{U}$ denotes the unit tangent bundle of $\bar{U}$. We see this by picking $(p, V) \in T^1\bar{U}$ and $q \in U$ then taking the parallel translation of V along the geodesic γ from p to q. Hence this space is compact and therefore there is a C such that $\left| \frac{d}{dt} \langle V(t), X_i \rangle \right| \leq C$. Hence C is a Lipschitz constant for $\langle V(t), X_i \rangle$ for any V, γ, i.

Now we claim that C is also a Lipschitz for γ that is not geodesic. For any curve γ, we can take an piecewise geodesic approximation $\bar{\gamma}$ of γ such that the parallel translation $\langle V(t), X_i \rangle$ and $\left\langle \bar{V}(t), X_i \right\rangle$ are within ϵ independent of t. By taking $\epsilon \to 0$, we have that $\langle V(t), X_i \rangle$ is also Lipschitz.

Now we argue that $\langle N, X_i \rangle$ is Lipschitz as a function of M, i.e. that if we define $f_i(p) = \langle N, X_i \rangle_p$, then $|f(p) - f(q)| \leq d(p, q)$ for all $p, q \in M$.

Fix points $p, q \in M$. Let $\gamma : [0, T] \to M$ be the geodesic from p to q parametrized by arc-length. Let $V_i = X_i$ at p and extend V_i as a parallel vector field along γ. By the previous proposition, $|\langle V_i(0), N \rangle - \langle V_i(T), N \rangle| \leq T \|V_i(0)\|$. Moreover, $\langle V_i(0), X_i \rangle = 0$ and $|\langle V_i(T), X_i \rangle| \leq CT$. Further note that $\langle N, V_j \rangle \leq 1$ since they are both unit vectors and that there exists an L such that $\langle X_i, X_i \rangle \leq L^2$ at all points of $\bar{U}$. Hence $|\langle V_j(T), X_i \rangle| \leq L$.

Denoting by $N(t)$ the vector N at $\gamma(t)$,

$$|\langle N(0), X_i\rangle - \langle N(T), X_i\rangle| = \left|\sum_{j=1}^{n} \langle N(0), V_j(0)\rangle \langle V_i(0), X_i\rangle - \langle N(T), V_j(T)\rangle \langle V_j(T), X_i\rangle\right| \tag{4.17}$$

$$\leq \sum_{j=1}^{n} \big| \langle N(0), V_j(0)\rangle \left(\langle V_j(0), X_i\rangle - \langle V_j(T), X_i\rangle\right) \tag{4.18}$$

$$+ \left(\langle N(0), V_j(0)\rangle - \langle N(T), V_j(T)\rangle\right) \langle V_j(T), X_i\rangle \big| \tag{4.19}$$

$$\leq n(CT + TL) \tag{4.20}$$

Since $T = d(p, q)$, we conclude that $\langle N, X_i\rangle$ is Lipschitz on M with constant $n(C + T)$.

Hence the Picard-Lindelöf existence theorem says that there exist solutions to the geodesic equation locally through any point in U. □

Proposition 23. *For any point $p \in \overline{M_C}$, there exists a C^1 integral curve γ of e_2 which is orthogonal to $\mathcal{F}$ at every point. If M is simply connected, then γ can be taken to be maximal in $\overline{M_C}$ and then intersects exactly once each plane in $\mathcal{F}$ in the connected component of $\overline{M_C}$.*

Proof. Since e_2 is orthogonal to $\mathcal{F}$ everywhere, Corollary 22 gives us existence of these integral curves through each point.

Assume next that M is simply connected, which gives uniqueness of geodesics since sec ≤ 0.

Now we consider the integral curve γ at some point $p \in \overline{M_C}$. We can assume that γ is maximal in the connected component V of $\overline{M_C}$ that contains p. Since γ has unit speed, the domain of $\gamma : I \to V$ is a closed interval I (possibly infinite or half-infinite). We claim that $\exp_\gamma^\perp$ is onto V, i.e. that γ intersects each leaf in $\mathcal{F}$ in V once. We first prove that $\mathrm{Im}(\exp_\gamma^\perp)$ is closed. If not, then there exists a point $q \in V$ and a sequence of points $q_k \in \mathrm{Im}(\exp_\gamma^\perp)$ with $q_k \to$. Then $\mathcal{F}_{q_k} \to \mathcal{F}_q$. For each q_k, let $\gamma(t_k)$ be the point on γ through $\mathcal{F}_{q_k}$. Since I is closed, if t_k is bounded, then the t_k have a limit point t_* in I, which implies that $\mathcal{F}_{t_*} = \mathcal{F}_q$,

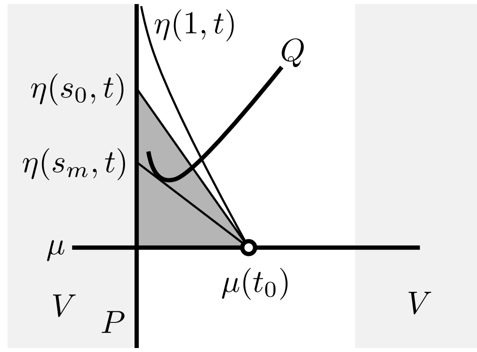

Figure 3: Setup of the contradiction in the proof of the convexity of V. The darker shaded region is the compact set over which we minimize s to find s_m so that $\eta(s_m,t)$ is tangent to Q giving the contradiction.

which is a contradiction. So we may assume that $t_k \to \infty$. Then $\mathcal{F}_{\gamma(t)} \to \mathcal{F}_q$ as $t \to \infty$.

Let η_t be the shortest path from $\gamma(t)$ to $\mathcal{F}_q$ and $y(t)$ be the length of η_t. Note that Proposition 21 gives that $\langle e_2, \eta_t' \rangle \geq 1 - y$. Considering the variation of geodesics η_x, the first arc-length variation formula says that

$$\frac{d}{dt}y = -\left\langle \gamma', \eta_t' \right\rangle = -\left\langle e_2, \eta_t' \right\rangle \leq -1 + y.$$

When $0 < y < 1/2$, we must have that $\frac{d}{dt}y < -1/2$, and hence $y(t) \to 0$ in finite time. (Geometrically, this says that since $\mathcal{F}$ is Lipschitz, a path orthogonal to $\mathcal{F}$ which is close to a plane in $\mathcal{F}$ must be approaching that plane nearly as quickly as possible.) Since I is closed, we again get a contradiction that $\mathcal{F}_q$ must be in $\exp_\gamma^\perp$. Hence $\mathrm{Im}(\exp_\gamma^\perp)$ is closed.

Now we want to show that $\mathrm{Im}(\exp_\gamma^\perp)$ is all of V. First we argue that V is convex. Refer to Figure 3 in this section. Suppose that there exists points $x_1, x_2 \in V$ such that the minimal geodesic $\mu : [a,b] \to \mathbb{R}$ between them is not contained in V. Then there exists a t_0 such that $\mu(t_0) \notin V$. Since V is closed, the set $S = \{t \in [a,b] | \mu(t) \in V\}$ is compact and since $t_0 \notin S$, $S \cap [a, t_0)$ and $S \cap (t_0, b]$ are compact as well. Hence there exists a $t_- = \max S \cap [a, t_0)$ and $t_+ = \min S \cap (t_0, b]$. Then $\mu((t_-, t_+))$ is disjoint from V.

Let $P = \mathcal{F}_{\mu(t_-)}$. Define U to be a subset of the unit vectors at $\mu(t_0)$ by

$$U := \{X \in T^1_{\mu(t_0)}M \,|\, \exp_{\mu(t_0)}(tX) \in P \text{ for some } t > 0\}. \tag{4.21}$$

Note that U is connected, and non-empty. It is open since $\exp_{\mu(t_0)}(tX)$ for $X \in U$ must be transverse to P since otherwise the fact that P is totally geodesic would imply that $\mu(t_0) \in P$. We claim that for $X \in \partial U$, $\exp_{\mu(t_0)}(tX)$ is disjoint from V for $t > 0$. This will allow U and $T^1_{\mu(t_0)}M \setminus \overline{U}$ to be two disjoint open sets which will force V to be disconnected. Take $X_0 \in \partial U$. Assume that for some $T > 0$, $q := \exp_{\mu(t_0)}(TX_0) \in V$. But q cannot be in P since then $X_0 \in U$ but U is open. Let Q be the plane of $\mathcal{F}_q$. Then the geodesic $\exp_{\mu(t_0)}(tX_0)$ is transverse from Q at q.

Choose a path $\alpha : [0,1] \to T^1_{\mu(t_0)}$ of unit vectors at $T_{\mu(t_0)}M$ such that $\alpha(0) = -\mu'(t_0)$ and $\alpha(1) = X_0$ and $\alpha(s) \in U$ for $s \in (0,1)$. Define $\eta(s,t) = \exp_{\mu(t_0)}(t\alpha(s))$. Since $\eta(1,t)$ is intersects Q transversely, for all s near 1, $\eta(s,t)$ must intersect Q for some T. Let s_0 be one such s near 1 such that the intersection of $\eta(s_0,t)$ with Q is also transverse. Then this also holds for s near s_0 and moreover for all $s \in [0,s_0]$, $\eta(s,t)$ intersects P.

Note that the length of each geodesic segment $t \mapsto \eta(s,t)$ ending at the point where it intersects P is bounded above by some length $L > 0$ for all $s \in [0,s_0]$. Restrict the domain of η to $\{s,t | s \in [0,s_0], \eta(s,t') \notin P \text{ for } t' < t\}$, i.e. so that t is before the intersection point with P. This domain is compact since the lengths of all such segments before intersecting P is bounded above. Then $\mathrm{Im}(\eta) \cap Q$ is compact since Q is closed. Hence we may define s_m to be the minimal s such that $\eta(s,t) \in Q$ for some point $\eta(s,t)$ in the compact domain. Since $\eta(0,t) = \mu(t+t_0)$ does not intersect Q before P and $\eta(s_0,t)$ intersects Q transversely, s_m must be in the interior $(0,s_0)$. Recall that Q is disjoint from P since they are distinct leaves in a foliation and moreover Q does not contain $\mu(t_0)$ since $Q \subset V$. Hence, the intersection point of Q with $\eta(s_m,t)$ must be in the interior of the compact domain.

Therefore $t \mapsto \eta(s_m,t)$ must be tangent to Q since otherwise there would be a smaller

$s < s_m$ that makes $\eta(s,t)$ intersect Q. This is a contradiction since $\mu(t_0) = \eta(s_0, 0)$ is not in Q but $t \mapsto \eta(s,t)$ is a geodesic and therefore contained in Q. Hence $\exp_{\mu(t_0)}(t\partial U)$ is disjoint from V for all t.

Hence we have two open, disjoint subsets of M defined as

$$\{\exp_{\mu(t_0)}(tX)|t > 0, X \in U\}, \quad \{\exp_{\mu(t_0)}(tX)|t > 0, X \in T^1_{\mu(t_0)} \setminus \overline{U}\}. \tag{4.22}$$

These cover V since $\exp_{\mu(t_0)}(tX)$ is never in V for $X \in \partial U$. This is a contradiction with the connectedness of V. Hence μ could not leave V and V must be convex.

This implies that V must equal $\mathrm{Im}(\exp^{\perp}_{\gamma})$. If not, then there would be a geodesic η in V which has points both in and not in $\mathrm{Im}(\exp^{\perp}_{\gamma})$. Then there is a point q on η on the boundary of $\mathrm{Im}(\exp^{\perp}_{\gamma})$ and η is transverse to $\mathcal{F}_q$. This implies the foliation $\mathcal{F}$ exists locally on both sides of q and hence on both sides of all points of $\mathcal{F}_q$. Hence $\mathcal{F}_q$ is in the interior of V and since we have already established that $\mathrm{Im}(\exp^{\perp}_{\gamma})$ is closed, there is a point of γ on $\mathcal{F}_q$. This is a contradiction, since γ was assumed to be maximal on V but γ could be extended past $\mathcal{F}_q$. Therefore we conclude that γ in fact intersects every leaf in the foliation that lies in the connected component V.

To see that γ intersects each leaf of the foliation exactly once, we will use that $\sec \leq 0$ and $\pi_1(M)$ is trivial. So for each point $\gamma(t_0)$, let P_0 be the leaf of $\mathcal{F}$ at $\gamma(t_0)$. Then there is some $\epsilon > 0$, so that the unique geodesic from $\gamma(t_0)$ to $\gamma(t)$ for $t \in (t_0, t_0 + \epsilon)$ does not lie in P_0.

Then we may define $L(t) > 0$ for each t so that $t + L(t)$ is the first time such that γ returns to the same leaf as $\gamma(t)$. Note that if $t_0 < t < t_0 + L(t_0)$, then $L(t) < L(t_0) - (t - t_0)$ since the leaf through $\gamma(t)$ cannot intersect P_0 and so γ must cross the leaf through $\gamma(t)$ twice before reaching P_0 the second time. This contradicts that $L(t) > 0$ for all t. Hence γ intersects each plane exactly once. $\square$

4.4. Locally Irreducible Metrics

In this section, we consider the case where M has Ricci eigenvalues $(-1,-1,0)$ and is locally irreducible at every point. Then there is no open set on which $C=0$. Hence $\overline{M_C}=M$ and so by Proposition 23, there exists a complete, unit speed curve $\gamma:\mathbb{R}\to M$ everywhere orthogonal to $\mathcal{F}$ so that each point of M lines on a plane of $\mathcal{F}$ through γ. Define $f(x):=a(\gamma(x))$ and $h(x)=\beta(\gamma(x))$, where $h(x)$ is defined only on the set $\{x\in\mathbb{R}: f(x)\neq 0\}$. We give a wide class of examples that show that h may not even extend continuously to all of $\mathbb{R}$.

Theorem 24. *Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth function and $\gamma:\mathbb{R}\to\mathbb{H}^2$ be a C^1 arc-length parametrized curve in $\mathbb{H}^2$ such that the geodesics orthogonal to γ are disjoint and cover $\mathbb{H}^2$. Define $S\subset\mathbb{R}$ to be the set of x such that $\gamma(x)$ is locally a smooth curve and define $h(x):=\left\langle\nabla_{\gamma'}\gamma',e_1\right\rangle$ on S (where e_1 is a unit normal vector field along γ). Suppose further that $f(x)$ satisfies that*

$$f^{(k)}(x)h^{(\ell_1)}(x)\cdots h^{(\ell_m)}(x)\to 0 \tag{4.23}$$

as $x\to x_0\notin S$, for any $k,m,\ell_1,\ldots,\ell_m\geq 0$.

Then there exists a complete metric g on $\mathbb{H}^2\times\mathbb{R}$ such that

(i) g has Ricci eigenvalues $(-1,-1,0)$,

(ii) g is of the form (4.13) in certain coordinates,

(iii) γ is orthogonal to $\mathcal{F}$,

(iv) $f(x)=a(\gamma(x))$, for all x,

(v) $h(x)=\beta(\gamma(x))$ for all $x\in S$, which satisfies $|h(x)|\leq 1$, and

(vi) M is locally irreducible if and only if there is no open subset on which f is zero.

First, we present non-trivial examples of the above.

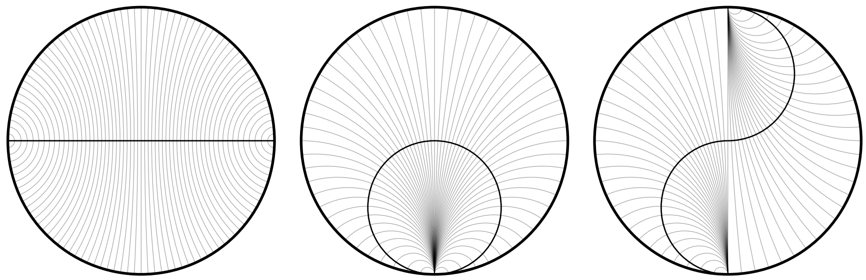

Figure 4: Three examples of possible choices of γ for Example 26, given as paths in the Poincaré disk model of $\mathbb{H}^2$.

Example 25. If $\gamma : \mathbb{R} \to \mathbb{H}^2$ is smooth, then there are smooth coordinates (x, u, v) on $\mathbb{H}^2 \times \mathbb{R}$ defined by $u(p)$ is the distance from p to γ, $x(p)$ is such that $\gamma(x(p))$ is the closest point on γ to p, and v gives the $\mathbb{R}$ factor. Then, for any choice of $f(x)$, the metric g is (4.13) on this choice of (x, u, v). Moreover, we have these solutions for any choice of $h : \mathbb{R} \to [-1, 1]$ smooth. We give characterisations of these curves in 4.4.1.

Example 26. In Figure 4, we see three examples of curves γ with their corresponding orthogonal geodesics, in the Poincaré disk model of $\mathbb{H}^2$. The first two are smooth curves, with $h(x) = 0$ and $h(x) = 1$. The last curve is only C^1 and has only one non-smooth point $\gamma(0)$. On the left half, it has $h(x) = 1$ and on the right $h(x) = -1$. Any choice of smooth $f(x)$ works for the first two examples. For the last example, any smooth $f(x)$ works so long as $f^{(k)}(0) = 0$ for all k.

Example 27. So long as γ is chosen so that its geodesic curvature $h(x)$ has bounded derivatives (on S), then the condition in Theorem 24 on $f(x)$ becomes that $f^{(k)}(x) = 0$ for $x \notin S$. The previous examples are special cases of this.

Proof. We proceed by defining g_f a smooth symmetric tensor on $M = \mathbb{H}^2 \times \mathbb{R}$ such that $g = g_{\mathbb{H}^2\times\mathbb{R}} + g_f$ is the desired metric. Note that we can embed γ into M by $(\gamma(x), 0)$, and we call this embedding γ as well for simplicity. Let e_1 be an unit vector field along γ which is orthogonal to γ' in $\mathbb{H}^2$, and let e_3 be a unit vector field in the $\mathbb{R}$ factor of M. There are C^0 coordinates (x, u, v) of M such that $p \in M$ has coordinates (x, u, v) if $p = \exp_{\gamma(x)}(ue_1 + ve_3)$. Define e_2 to be a unit vector field parallel to orthogonal to $\{e_1, e_3\}$.

Let $S \subset \mathbb{R}$ be the set of x values such that γ is locally smooth at $\gamma(x)$. Then there is a subset $S_M \subset M$ of points p such that $x(p) \in S$. Then S_M is the set of points where the (x, u, v) coordinates are locally smooth. We also consider S_M as a subset of $\mathbb{R}^3$ by $(x, u, v) \in S_M$ whenever $p = (x, u, v)$ is in $S_M \subset \mathbb{H}^2 \times \mathbb{R}$.

Note that M has Ricci eigenvalues $(-1, -1, 0)$ with $C = 0$ and on S_M has a smooth foliation by complete geodesic planes and hence the metric is of the form (4.13) on S_M by Proposition 18. Therefore, the vector $e_3 = T$ and $\{e_1, e_2, T\}$ satisfy all the equations of (4.4)-(4.8) when taking covariant derivatives with the Levi-Civita connection of $g_{\mathbb{H}^2\times\mathbb{R}}$ where $f(x) = 0$ in those equations and $h(x) := \langle \nabla_{\gamma'}\gamma', e_1 \rangle$. Moreover, the contents of Remark 19 also apply in S_M. (Our goal is to modify $g_{\mathbb{H}^2\times\mathbb{R}}$ to make our choice of $f(x)$ the one that occurs in these covariant derivatives.)

Define the symmetric 2-tensor g_f point-wise on M at points $p \in S_M$ by

$$g_f = -2f(x)v(dx\ du + du\ dx) + 2f(x)u(dx\ dv + dv\ dx) + f(x)^2(u^2 + v^2)dx^2 \tag{4.24}$$

and by $g_f = 0$ for $p \notin S_M$. Using that $e_1 = \frac{\partial}{\partial u}$, $e_2 = \frac{\partial}{\partial x}(\cosh u - h(x)\sinh u)^{-1}$, and $e_3 = \frac{\partial}{\partial v}$, we get that

$$g_f(X_1, X_2) = -2f(x)(\cosh u - h(x)\sinh u)^{-1}v\left(\langle X_1, e_1\rangle \langle X_2, e_2\rangle + \langle X_1, e_2\rangle \langle X_2, e_1\rangle\right) \tag{4.25}$$

$$+ 2f(x)(\cosh u - h(x)\sinh u)^{-1}u\left(\langle X_1, e_3\rangle \langle X_2, e_2\rangle + \langle X_1, e_2\rangle \langle X_2, e_3\rangle\right) \tag{4.26}$$

$$+ f(x)^2(\cosh u - h(x)\sinh u)^{-2}(u^2 + v^2)\langle X_1, e_2\rangle \langle X_2, e_2\rangle \tag{4.27}$$

where $\langle \cdot, \cdot \rangle$ is the inner product with respect to $g_{\mathbb{H}^2\times\mathbb{R}}$.

Let Φ be a set of smooth functions $\phi(x,u,v)$ on S_M defined by

$$\Phi := \{u, v, \cosh u, \sinh u\} \cup \{f^{(i)}(x)\}_{i\geq 0} \cup \{h^{(i)}(x)\}_{i\geq 0} \tag{4.28}$$
$$\cup \{\langle X, e_i\rangle \,|X \text{ any smooth vector field on } M \text{ and any } i\}. \tag{4.29}$$

For a smooth function $F(x,u,v)$ on the domain S_M, we consider three properties:

(A) that F is a rational function of some functions in Φ,

(B) that F satisfies (A) and its denominator is bounded away from 0 on the set where $|u| \leq R$ for any $R \geq 0$, or

(C) that F satisfies (A) and each term of its numerator has a positive power $f^{(i)}(x)$ for some $i \geq 0$.

We say that F is an ABC function if it satisfies (A), (B), and (C) and similarly say F is an AB function if it satisfies (A) and (B) but not necessarily (C).

By (4.25) we know can see that $g_f(X_1, X_2)$ is an ABC function for any X_1, X_2 smooth vector fields on M. We will show that if $F(x,u,v)$ is an ABC function, then $X_1 \cdots X_k(F)(x,u,v)$ is an ABC function as well and that any ABC function goes to zero on $M \setminus S_M$. We will then conclude by showing that this implies that all ABC functions extend smoothly to all of M with $F = 0$ on $M \setminus S_M$.

First, observe that $a(x,u,v) = f(x)/\left(\cosh u + h(x)\sinh u\right)$ is an ABC function since

$$|\cosh u + h(x)\sinh u| \geq e^{-u} \tag{4.30}$$

$|h(x)| \leq 1$. Similarly, $\beta(x,u,v) = (h(x)\cosh u - h(x)\sinh u)(\cosh u - h(x)\sinh u)^{-1}$ is an AB function (but not necessarily an ABC function). Hence, by the computations of (4.4)-(4.8), any $\langle \nabla_{e_i} e_j, e_k\rangle$ is an AB function (where ∇ denotes the covariant derivative with respect to $g_{\mathbb{H}^2\times\mathbb{R}}$). Any $\phi \in \Phi$ is AB as well. Moreover, note that the product of an ABC function with an AB function is an ABC function.

We next show that if $\phi \in \Phi$, then $X(\phi)$ is also an AB function for any smooth vector field X on M. Write $X = \sum_{i=1}^{3} \langle X_1, e_i \rangle e_i$. So it suffices to consider $e_i(\phi)$. For $\phi = \langle Y, e_k \rangle$, we can compute that

$$e_i \langle Y, e_k \rangle = \langle \nabla_{e_i} Y, e_k \rangle + \langle Y, \nabla_{e_i} e_k \rangle = \sum_{j=1}^{3} \langle \nabla_{e_i} e_j, e_k \rangle \langle Y, e_j \rangle + \langle Y, e_j \rangle \langle \nabla_{e_i} e_k, e_j \rangle \tag{4.31}$$

which is an AB function.

If ϕ is not $\langle Y, e_k \rangle$, then $\frac{\partial}{\partial x}(\phi), \frac{\partial}{\partial u}(\phi), \frac{\partial}{\partial v}(\phi)$ are all in Φ and hence are AB. Hence $e_1(\phi)$ and $e_3(\phi)$ are immediately AB (since $e_1 = \frac{\partial}{\partial u}$ and $e_3 = \frac{\partial}{\partial v}$). For $e_2(\phi)$, on S_M, we have $e_2 = (\cosh u - h(x) \sinh u)^{-1} \frac{\partial}{\partial x}$. Since $(\cosh u - h(x) \sinh u)^{-1}$ is an AB function and we can conclude that $e_i(\phi)$ is AB for any $i = 1, 2, 3$. Hence $X(\phi)$ is an AB function for any smooth vector field X and $\phi \in \Phi$.

This shows that if F is an AB function then $X(F)$ is an AB function since $F = P/Q$ with P and Q rational functions of the functions in Φ and hence $X(F)$ is rational in the functions of Φ with denominator Q^2. Since Q is bounded away from zero, Q^2 is as well and $X(F)$ is an AB function. Moreover, if F is an ABC function, then $X(F)$ is also an ABC function. This follows from the fact that that $\frac{\partial}{\partial u}F, \frac{\partial}{\partial v}F$, and $\frac{\partial}{\partial x}F$ are all ABC functions if F is (since an u, v, or x derivative of $f^{(i)}(x)$ either takes the term to zero or leaves a factor $f^{(i+1)}(x)$). Hence $X_1 \cdots X_n(F)$ is an ABC function for any ABC function F.

Suppose that F is an ABC function. We wish to show that F extends continuously to all of M with $F = 0$ at points in $M \setminus S_M$. Fix $R > 0$. Then any $\phi \in \Phi$ other that $\phi = h^{(i)}$ is bounded on $B_R(0) \cap S_M$ since $f(x)$ is a smooth function of x, and $\langle X, e_i \rangle \leq \|X\|$ which is bounded since X is smooth. Suppose that (x_k, u_k, v_k) is a sequence in $B_R(0) \cap S_M$ that converges to some point not in S_M. Then x_k converges to a point not in S. Therefore $f^{(i)}(x_k) h^{(j_1)}(x_k) \cdots h^{(j_m)}(x_k) \to 0$ as $k \to \infty$ by our assumption on f. Since F is an ABC function, all terms involve some $f^{(i)}(x)$ factor and all factors other than $h^{(j)}(x)$ are bounded, so the numerator of F goes to zero. Since the denominator of F is bounded for $u \in B_R(0)$, we have that $F(x_k, u_k, v_k) \to 0$ as $k \to \infty$.

We then claim that an ABC function F must be smooth on M with all derivatives 0 on $M \setminus S_M$. Note that F is smooth on S_M, so it remains to show that F is smooth on $M \setminus S_M$. It suffices to show that F is C^1 since any derivative of F is also an ABC function and hence would also be C^1. So we need to show that the first partials of F exist and are continuous, hence we must show that they are 0 on $M \setminus S_M$. Consider a point $p_0 = (x_0, u_0, v_0) \in M \setminus S_M$. Let η be any path with $\eta(0) = p_0$. We want to see that $\frac{d}{dt} F(\eta(t)) = 0$ at $t = 0$. Since $F(\eta(0)) = 0$, we do this by showing that $F(\eta(t))$ is $o(|t|)$. If $\eta(t_1) \notin S_M$, then $F(\eta(t_1)) = 0$ and we are done. If not, suppose $t_1 > 0$, and then we can pick $0 \leq t_0 < t_1$ such that η is in S_M on (t_0, t_1) and $\eta(t_0) \notin S_M$. Then $F \circ \eta(t_1) = \int_{t_0}^{t_1} (\eta'(F))(\eta(t))dt$. Since $X(F)$ is an ABC function as well, we know that $X(F)(\eta(t)) \to 0$ as $t \to 0$ and hence we can choose $\delta > 0$ such that $|X(F)| \leq \epsilon$ at all points $\eta(t)$ with $|t| \leq \delta$. Hence $F \circ \eta(t_1) \leq \epsilon\,|t_1 - t_0| \leq \epsilon\,|t_1|$. Choosing $X = \eta'$ and taking $\epsilon \to 0$, we see that $F \circ \eta$ is $o(|t|)$. Hence the partial derivative of F in the direction of $\eta'(0)$ is well-defined and therefore F is C^1 on all of M. Hence F is C^∞ on M.

This implies, in particular, that $g_f(X, Y)$ is a smooth function for X, Y fixed. Bilinearity of g_f point-wise means that g_f is a smooth tensor on M.

Therefore $g = g_{\mathbb{H}^2 \times \mathbb{R}} + g_f$ is smooth and is of the form (4.13) on $\tilde{S}$. On $M \setminus \tilde{S}$, $g = g_{\mathbb{H}^2 \times \mathbb{R}}$ and hence M has Ricci eigenvalues (-1,-1,0) everywhere. $\square$

Remark 28. Finally, we note that for each complete, simply connected M with Ricci eigenvalues $(-1, -1, 0)$ that is locally irreducible everywhere, there exists a $\gamma : \mathbb{R} \to M$ orthogonal to $\mathcal{F}$ and $f(x) := a(\gamma(x))$. The C^0 coordinates (x, u, v) defined by $p = (x, u, v)$ if $p = \exp_{\gamma(x)}(ue_1 + vT)$ has g of the form (4.13) on the dense, open region where γ is smooth. On each connected component where $\mathcal{F}$ is smooth, we can modify g by setting $f(x) = 0$ which makes the metric isometric to the product metric $\mathbb{H}^2 \times \mathbb{R}$ with γ contained in $\mathbb{H}^2 \times \{0\}$. This gives a candidate for a converse to Theorem 24: for each such M there exists an γ, f.

However, it is not clear that such an f always satisfies the assumption on it in Theorem 24.

Below we give a partial converse to the above theorem for the case when γ and $\mathcal{F}$ are smooth.

Theorem 29. *Suppose M is complete and simply connected and has Ricci eigenvalues $(-1,-1,0)$. If M is everywhere locally irreducible and $\mathcal{F}$ is smooth on M, then M has smooth coordinates (x,u,v) such that*

$$g = (\cosh u - h(x)\sinh u)^2 dx^2 + (du - f(x)v\ dx)^2 + (dv + f(x)u\ dx)^2 \tag{4.32}$$

for some smooth functions $f,h : \mathbb{R} \to \mathbb{R}$ with $|h| \leq 1$.

Proof. Since $\mathcal{F}$ is smooth, the (x,u,v) coordinates are smooth, the vector fields e_1, e_2, T are smooth, and there is a smooth curve γ orthogonal to every leaf of $\mathcal{F}$. Hence $a = -\langle \nabla_{e_2} T, e_1 \rangle$ and $\beta = \langle \nabla_{e_2} e_2, e_1 \rangle$ are smooth functions. Therefore $f(x) := a(\gamma(x))$ and $h(x) := \beta(\gamma(x))$ are smooth functions $\mathbb{R} \to \mathbb{R}$. Hence the right hand side of (4.32) defines a smooth tensor.

From Proposition 18, we know that the metric g satisfies (4.32) on each component of M_C. Since M is locally irreducible, M_C is a dense set, and hence g satisfies (4.32) everywhere. □

4.4.1. Foliating Curves

We now present some properties of the curves γ in Theorem 24. Suppose that $\gamma : \mathbb{R} \to \mathbb{H}^2$ is a C^1 curve that is arc-length parameterized. Let X be a unit vector field along γ that is perpendicular to γ' everywhere. Define $\exp^\perp : \mathbb{R}^2 \to \mathbb{H}^2$ by

$$\exp^\perp(s,t) = \exp_{\gamma(s)}(tX). \tag{4.33}$$

Proposition 30. *The following are equivalent.*

(I) The geodesics $\eta_s : t \mapsto \exp^\perp(s,t)$ form a foliation of $\mathbb{H}^2$.

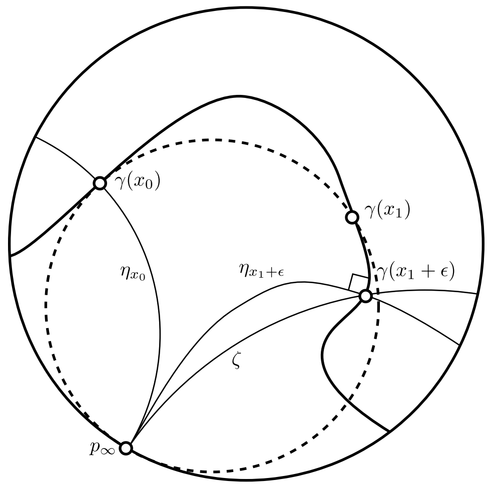

Figure 5: Diagram of the proof that (I) implies (II), considered in the Poincaré disk model.

(II) γ *is disjoint from all (open) horoballs tangent to* γ*.*

If γ *is further assumed to be* C^∞*, then the following are also equivalent to the above.*

(III) $\left|\nabla_{\gamma'}\gamma'\right| \leq 1$ *at all points.*

(IV) γ *has no focal points.*

Proof. We start with showing that (I) implies (II). Suppose that (II) does not hold. Then there are $x_0 < x_1 \in \mathbb{R}$ and H_0 one of the two horoballs tangent to γ at $\gamma(x_0)$ such that $\gamma(x_1)$ is on the boundary of H_0 and $\gamma(x_1+\epsilon)$ is in H_0 for all $\epsilon > 0$. I.e. x_1 is the point where γ crosses into a horoball of at x_0.

Consider B, the Busemann function of the ray η_s (where we may assume that η_s is such that it points in the direction of H_0). Let p_∞ be the ideal point along that ray. Then $B(\gamma(x_i)) = 0$ as well but $B'(\gamma(s)) > 0$ at $x_1+\epsilon$ for small $\epsilon > 0$. Recall that the gradient ∇B at a point p is always tangent to the geodesic from p to the ideal point p_∞. Let ζ be the geodesic at $\gamma(x_1+\epsilon)$ (for a fixed small $\epsilon > 0$) to the ideal point p_∞. Then $\langle \zeta', \gamma' \rangle > 0$ at $\gamma(x_1+\epsilon)$ since $B'(\gamma(x_1+\epsilon)) > 0$.

By (I), we know that $\eta_{x_1+\epsilon}$ intersects neither η_{x_0} nor γ except at $\gamma(x_1+\epsilon)$. Hence one half

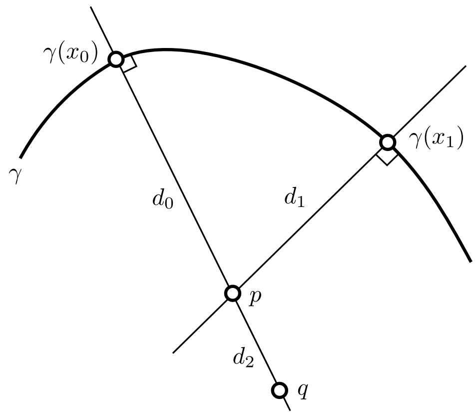

Figure 6: Diagram of the proof that (II) implies (I).

of $\eta_{x_1+\epsilon}$ must be in the region bounded by η_{x_0}, γ, and ζ. See Figure 5. In particular, the ideal point of that half of $\eta_{x_1+\epsilon}$ must be between the ideal points of η_{x_0} and ζ, both of which are p_∞. Hence p_∞ is an ideal point on $\eta_{x_1+\epsilon}$. But ζ is the unique geodesic from $\eta_{x_1+\epsilon}$ to p_∞ and $\langle \zeta', \gamma' \rangle > 0$ implies that $\eta_{x_1+\epsilon}$ cannot be ζ. Hence there is a contradiction.

Next we show that (II) implies (I). Suppose that (I) does not hold. Then there is $x_0 \neq x_1 \in \mathbb{R}$ such that η_{x_0} intersects η_{x_1} at a point p. Suppose without loss of generality that $\gamma(x_1)$ is at least as close to p as $\gamma(x_0)$ is to p. Then let $d_i = d(p, \gamma(x_i))$ so that $d_1 \leq d_0$. Pick a point q on η_{x_0} on the far side of p from $\gamma(x_0)$. Let $d_2 = d(p, q)$. See Figure 6.

Since η_{x_1} is not the same geodesic as η_{x_0}, there must be a path from q to $\gamma(x_1)$ that is strictly shorter than the piecewise geodesic path from q to p and then to $\gamma(x_1)$. Hence

$$d(q, \gamma(x_1)) \leq d_1 + d_2 - \epsilon < d_2 + d_0 = d(q, \gamma(x_0)). \tag{4.34}$$

Hence $\gamma(x_1)$ is in the horoball tangent to γ at $\gamma(x_0)$. This contradicts (II).

For the remainder, we assume that γ is C^∞. We also compute the Jacobi field of the variation of geodesics $s \mapsto \eta_s$ about some $\gamma(x_0)$ as $J(t) = \left[\cosh(t) - (\nabla_{\gamma'}\gamma'(x_0)) \sinh t\right] e_2$ where e_2 is the unit vector field orthogonal to each η_s with $e_2 = \gamma'$ on γ. This calculation shows that (III) and (IV) are equivalent since $\cosh t - h \sinh t$ has a zero for some $t \in \mathbb{R}$ if and only if $|h| \geq 1$.

Moreover, we can easily see that (I) implies (III). Suppose that (III) does not hold. Then there is some x_0 such that $\left|\nabla_{\gamma'}\gamma'\right| > 1$ at $\gamma(x_0)$. Hence the Jacobi field $J(t)$ at η_{x_0} has a zero and moreover $\langle J(t), e_2\rangle$ changes sign at some point. Since $\mathbb{H}^2$ is a 2-manifold, we must have that $\eta_{x_0+\epsilon}$ intersects η_{x_0} for every small ϵ. This contradicts (I).

Lastly, we see that (IV) implies (I). Since there are no focal points, $\exp^{\perp}$ is a local diffeomorphism $\exp^{\perp} : \mathbb{R}^2 \to \mathbb{H}^2$. We claim that $\exp^{\perp}$ is in fact a covering map. Let $g^* = (\exp^{\perp})^*(g)$ be the pullback metric on $\mathbb{R}^2$ the perpendicular tangent space to γ. Then $\exp^{\perp} : (\mathbb{R}^2, g^*) \to (\mathbb{H}^2, g)$ is a local isometry and hence it suffices to show that $(\mathbb{R}^2, g^*)$ is complete to see that $\exp^{\perp}$ is a covering map. Let (s,t) be the standard coordinates on $\mathbb{R}^2$.

Let e_1, e_2 be unit C^1 vector fields on $\mathbb{H}^2$ such that e_1 is parallel to the geodesic foliation and e_2 is orthogonal. Since (IV) is equivalent to (III), we can use $\left|\nabla_{\gamma'}\gamma'\right| \leq 1$ to give the lower bound $e^{-t} \leq \cosh(t) - (\nabla_{\gamma'}\gamma'(x_0))\sinh t$. By the computation of $J(t)$ above, we get that

$$\tfrac{\partial}{\partial s}\exp^{\perp}(s,t) = \left[\cosh t - (\nabla_{\gamma'}\gamma')\sinh t\right] e_2$$

and it is immediate that $\frac{\partial}{\partial t}\exp^{\perp}(s,t) = e_1$. By the lower bound above, we get that the metric $g' := e^{-2|t|}ds^2 + dt^2$ satisfies $g^* \geq g'$. So it suffices to show that g' is complete to see that g^* is complete.

We can see metric completeness of g' by considering a Cauchy sequence of points (s_k, t_k) satisfying $g'((s_k,t_k),(s_{k+\ell},t_{k+\ell})) \leq 2^{-k}$ for $\ell > 0$. Then $|t_k - t_0| \leq 2^{-k} \leq 1$ and hence $|s_k - s_0| \leq e^{2(t_0+1)}$. In this region $\{(s,t) \,|\, |s-s_0| \leq e^{2(t_0+1)}, |t-t_0| \leq 1\}$ we have that $g' \geq e^{-2|t_0+1|}g_{\mathbb{R}^2}$ and hence the Cauchy sequence in g' is also a Cauchy sequence under the standard Euclidean metric $g_{\mathbb{R}^2}$ of $\mathbb{R}^2$. Hence $(\mathbb{R}^2, g^*)$ is complete and $\exp^{\perp}$ is a covering map. Since $\mathbb{H}^2$ is simply connected, $\exp^{\perp}$ is then a diffeomorphism. Hence the geodesics orthogonal to γ form a foliation of M. $\square$

4.5. Manifolds with Locally Reducible Points

Now we want to describe the structure of complete, simply connected manifolds M which have Ricci eigenvalues $(-1, -1, 0)$ that may not be locally irreducible everywhere.

Theorem 31. *Suppose that a complete, simply connected manifold M has Ricci eigenvalues $(-1, -1, 0)$. Then M is decomposed as a union of disjoint regions $\{U_i\}$ such that,*

- *each U_i is either an open connected component of the interior of M_0 or is a closed connected component of $\overline{M_C}$,*
- *in the first case, we call U_i a* split region *and U_i is isometric to $\Sigma \times \mathbb{R}$ for $\Sigma \subset \mathbb{H}^2$ a connected subset of the hyperbolic plane whose boundary components are complete geodesics,*
- *in the second, we call U_i a* non-split region *and U_i has every point locally irreducible with $C \neq 0$ on a dense, open subset, are foliated by $\mathcal{F}$, and have a path γ_i orthogonal to $\mathcal{F}$ which intersects every leaf once.*

Proof. For each connected component of the interior of M_0 and each connected component of $\overline{M_C}$, we have a U_i. Since $M \setminus \overline{M_C}$ is the interior of M_0, we have that M is the union of these disjoint sets. If U is a non-split region, then its structure is given by Proposition 23.

Consider U a split region, so $C = 0$ on U. By the de Rham-type splitting result of [FZ16, PR93], we know that U_i is isometrically the product of $\Sigma \times \mathbb{R}$ for some surface Σ with Gaussian curvature -1. Each boundary component of U must also be a boundary component of a non-split region. Since non-split regions have complete, flat, totally geodesic boundary components, so too must U. Since M is simply connected, Lemma 17 says that each U is simply connected. Hence Σ is simply connected. To see that $\Sigma \subset \mathbb{H}^2$, we can consider its double $\Sigma \cup \Sigma$ glued along the geodesic boundary components. This is a complete surface with $K = -1$ and hence its universal cover is $\mathbb{H}^2$. Since Σ is simply connected, its inclusion into the double then lifts to an inclusion in $\mathbb{H}^2$, as desired. □

Next, we introduce a simplifying assumption which will be necessary to study the fundamental group.

Definition 32. If M has Ricci eigenvalues $(-1,-1,0)$, then we say that M has *locally finite* split regions if each compact subset of M intersects at most finitely many split regions. Hence, each compact region also intersects at most finitely many non-split regions.

Theorem 33. *Suppose that a complete, simply connected manifold M has Ricci eigenvalues $(-1,-1,0)$ and has locally finite split regions. Then we may associate to M a tree Γ_M which has nodes U_i given by the split and non-split regions of M and edges connected U_i to U_j if U_i and U_j share a boundary component. Moreover, if U is a non-split region, then it has at most two edges, and each edge is between a non-split region and split region.*

Proof. It is clear that we can associate a graph to M with the specified nodes and edges. That each non-split region U has at most two edges follows from the fact that the boundaries of the non-split regions are the planes at the boundary points of any curve orthogonal to $\mathcal{F}$. Each split region must share boundary components only with non-split regions (and vice-versa) because they are each the connected components of $M \setminus \overline{M_C}$ or $\overline{M_C}$.

We next claim that the manifold M deformation retracts to this graph and therefore Γ_M is a tree. We can realize this graph embedded within M by picking for vertices one point x_i in each split U_i. For U_i split, pick geodesics $\gamma_{i,j}$ from x_i to the closest point on each boundary component to x_i. For U_j non-split, if there are two split regions U_{i_1}, U_{i_2} bordering U_j, then take γ_j to be the geodesic connecting the endpoints of $\gamma_{i_1,j}$ and $\gamma_{i_2,j}$ that lie in U_j. The union of these paths is Γ_M.

Now we give a homotopy equivalence of M and Γ_M. Note that γ_j is transverse to $\mathcal{F}$ since each leaf of $\mathcal{F}$ is totally geodesic and γ_j starts transverse to the boundary leaves. Hence, on each non-split U_i, we can use (x,u,v) coordinates defined by $p=(x,u,v)$ if $p=\exp_{\gamma(x)}(ue_1+vT)$. For the map $\phi_t : M \to \Gamma_M$, take each point $(x,u,v) \in U_i$ to (x,tu,tv) if U_i is non-split. If U_i is a split region, then take each point $x \in U_i$, let η_x be the geodesic from x to the closest point on any $\gamma_{i,j}$ with $\eta_x(1)=x$ and $\eta_x(0)$ on $\gamma_{i,j}$. Then let

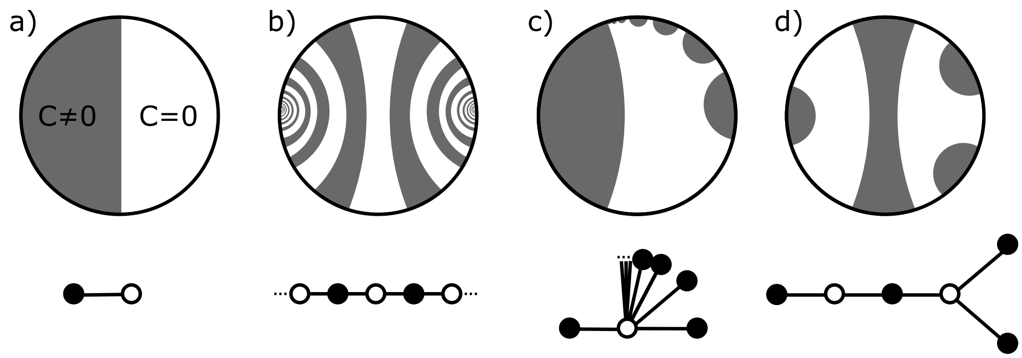

Figure 7: Examples of the trees constructed in Theorem 33.

$\phi_t(x) = \eta_x(t)$. Then ϕ_t continuous on each U_i region, for $t \in [0,1]$. Since each $\gamma_{i,j}$ meets the boundary plane between U_i and U_j orthogonally, ϕ_t is continuous at the boundary as well. Moreover, it depends continuously on t and ϕ_1 is the identity while ϕ_0 maps M onto Γ_M.

Therefore $\pi_1(M) = \pi_1(\Gamma_M)$ and since M is simply connected, Γ_N is a tree. □

Example 34. Figure 7 shows four possibilities for Γ_M. Each M is drawn schematically in the Poincaré disk model of $\mathbb{H}^2$ where the split regions are left white and the non-split regions are shaded. Each non-split region may have any number of boundary components, including infinitely many. Note that each split region has at most two (and possibly only one) boundary component. We can construct these examples by taking non-split regions of the form (4.13) with $f(x) = 0$ outside of some interval. Then these metrics are split outside of a strip and hence can be glued along their split regions.

4.6. Isometries and the Fundamental Group

In this section, we consider M with Ricci eigenvalues $(-1, -1, 0)$ that may not be simply connected. Since $\sec \leq 0$, the universal cover $\widetilde{M}$ is always diffeomorphic to $\mathbb{R}^3$ and the topology of M is determined by the fundamental group alone.

We have seen in Theorem 33 that $\widetilde{M}$ has an associated graph Γ_M whose nodes are the split and non-split regions. Then any isometry of $\widetilde{M}$ must induce an isometry of Γ_M and, moreover, must take non-split regions to non-split regions and split regions to split regions.

Our main result in this section is the following theorem.

Theorem 35. *Suppose that M is complete and has Ricci eigenvalues $(-1,-1,0)$ with locally finite split regions. If M has some locally irreducible points and some locally reducible points, then $\pi_1(M)$ is free. If M is locally irreducible at all points, then $\pi_1(M)$ is either trivial or $\mathbb{Z}$.*

We will prove four lemmas first, then use Bass-Serre theory to prove the theorem with the help of these lemmas. Let $\widetilde{M}$ be the universal cover of M. First, we recall that since sec ≤ 0, if G acts on $\widetilde{M}$ fixed point freely, then G cannot have torsion. If it did have torsion, then some $g \in G$ has $g^k = e$ for some k and so every orbit of $\langle g \rangle$ on $\widetilde{M}$ is finite and hence the centroid of the orbit is fixed by $\langle g \rangle$. Hence g must have been e itself. This then proves that if $g \in G$ has an invariant plane, i.e. $g(P) \subset P$ for some plane P in $\mathcal{F}$, then we may assume that g acts by translations on the plane. Certainly g acts by isometries on P and has no fixed point and hence is either a translation or a glide reflection. In the latter case, g^2 is a translation so we may pass to g^2 instead of g if necessary and showing g^2 is trivial shows that g itself is trivial. Moreover, if it fixes a finite number of planes, then we may assume it acts by translations on all of them at once. We use this assumption in the following lemmas.

Lemma 36. *Suppose V is a non-split region of $\widetilde{M}$. If G acts on $\widetilde{M}$ by isometries fixed point freely and $g(V) \subset V$ for all $g \in G$, then G is trivial.*

Proof. Let P_0, P_1 be the two boundary components of V. Observe that $g \in G$ must either preserve the boundary planes, $g(P_0) \subset P_0$ and $g(P_1) \subset P_1$, or it swaps them, $g(P_0) \subset P_1$ and $g(P_1) \subset P_0$. If g swaps the boundary planes, then g^2 must preserve them and $g^2 = e$ implies that $g = e$ by our previous observation, so we may assume that g preserves the boundary planes.

Now we will show that the only isometry in G that preserves two distinct planes is the identity. Assume that $g \neq e$. First note that $g|_{P_i}$ is an isometry of the flat plane $P_i \simeq \mathbb{R}^2$.

Since g has no fixed points, we may assume it acts by translation on each P_i. Pick a point $p_0 \in P_0$ and let γ_0 be the geodesic along which g translates P_0. Recall P_1 is a totally geodesic plane, and hence is a convex subset of $\widetilde{M}$. Since $\sec \leq 0$, we have that $d(\cdot, P_1)$ is a convex function along any geodesic and in particular along γ_0.

Moreover, since g acts by isometries and leaves P_1 invariant, $d(g^k(p_0), P_1) = d(p_0, P_1)$ and hence $d(\cdot, P_1)$ is constant along γ_0. Let p_1 be the unique point on P_1 closest to p_0 and let γ_1 be the geodesic along which g translates p_1. Then $g^k(p_1)$ is the unique point on P_1 closest to $g^k(p_0)$. Then γ_0 and γ_1 are parallel in the sense of having bounded (in fact constant) distance. Recall that, since $\sec \leq 0$, the union of all geodesics parallel to any geodesic is a closed convex subset isometric to $N \times \mathbb{R}$ for some closed convex subset N of M. See Lemma 2.4 in [BGS85]. Then the geodesics γ_0 and γ_1 bound an flat strip, i.e. a totally geodesic subset isometric to $[0, \ell] \times \mathbb{R}$.

Since this strip is flat, it must contain the nullity geodesics through each point of the strip. Since the nullity geodesics are complete, we must then have that they are parallel in the strip. Since this strip is totally geodesic and starts transverse to P_0, it must always be transverse to the vector field e_1. Hence T is parallel in every direction and the splitting tensor is identically 0 on the strip. This is contradiction with the assumption that P_0 and P_1 bounded a component where $C \neq 0$ in an open dense set. □

Lemma 37. *Suppose G acts on M by isometries and $G(V) \subset V$ for some V a non-split region exactly one boundary component. If G acts fix point freely, then G is trivial.*

Proof. Let P_0 be the unique plane of V. Since boundary planes are isometry invariants, $G(P_0) \subset P_0$.

Now let γ be a geodesic orthogonal to P_0 with $\gamma : [0, \infty) \to V$. Label the planes of $\mathcal{F}$ by P_x for P_x the unique plane through $\gamma(x)$. Recall that the planes of $\mathcal{F}$ are isometry invariants and hence g acts by $g(P_x) = P_{\bar{g}(x)}$ for some function $\bar{g} : [0, \infty) \to [0, \infty)$. By the previous lemma, the only $g \in G$ that fixes a second plane P_{x_0} is the identity. So $\bar{g}$ is a continuous bijection and which has a unique fixed-point 0. Hence (for $x > 0$), $\bar{g}$ is either strictly monotone increasing or strictly monotone decreasing. Assume without loss of generality that $\bar{g}$ is monotone decreasing (if not, then replace g with g^{-1}). Since there are no other fixed points than 0, the sequence $\{\bar{g}^k(x)|k > 0\}$ must go to zero for all $x > 0$.

For each $p \in V$, we take γ_p to be the unique C^1 integral curve orthogonal to $\mathcal{F}$ from P_0 to p. Specifically, we need $\gamma_p(0) \in P_0$, $\gamma_p(x_p) = p$ for some x_p, and $\gamma'(x_p) = e_2$. Now define $A(p) = \int_0^{x_p} |a(\gamma(x))| \, \|\gamma'(x)\| \, dx$. First observe that since $g \in G$ satisfies $g(P_0) \subset P_0$, A is invariant under g.

Next, we claim that A is constant on each leaf of $\mathcal{F}$. We compute the integral in explicit coordinates on the regions where $\mathcal{F}$ is smooth. On any connected component of the set where $a \neq 0$, we have explicit coordinates $(x, u, v) \in (x_0, x_1) \times \mathbb{R}^2$ such that

$$g = C(x)^2(\cosh u - h(x)\sinh u)^2 dx^2 + (du - v\ dx)^2 + (dv + u\ dx)^2 \tag{4.35}$$

for some $C(x) \neq 0$, $|h(x)| \leq 1$. Recall that in these coordinates, since $T = \frac{\partial}{\partial v}, e_1 = \frac{\partial}{\partial u}$, the vector e_2 is proportional to $X := \frac{\partial}{\partial x} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$ and $\|X\| = |C(x)| \, (\cosh(u) - h(x)\sinh u)$ and the leaves of $\mathcal{F}$ are the sets where x is constant. Moreover, $a = 1/(C(x)(\cosh u - h(x)\sinh u))$ and hence $\|X\| = 1/a$.

Then we have that the integral curves of e_1 are $\gamma(x) = (x, r_0 \sin(t + \theta_0), r_0 \cos(t + \theta_0))$ with $\gamma' = X$. This gives that

$$\int_{x_0}^{x} |a(\gamma(t))| \, \|\gamma'(t)\| \, dt = \int_{x_0}^{x} dt = x - x_0. \tag{4.36}$$

Hence this integral depends only upon the x-coordinate and hence two integral curves γ_0, γ_1 ending at the same leaf of $\mathcal{F}$ will have $\int a \, \|\gamma_i\| \, dt$ the same.

This then implies that A is constant on the leaves of $\mathcal{F}$ since $\int_0^x a(\gamma(t)) \, \|\gamma'(t)\| \, dt$ can be computed just on the domain where $a \neq 0$ and hence on a series of domains exactly as above. Note that $A = 0$ on P_0 and so the fact we proved above that $\bar{g}^k(x) \to 0$ as $k \to 0$ says that g^k takes any leaf of $\mathcal{F}$ to P_0 as $k \to \infty$. Hence $A(g^k(p)) \to 0$ as $k \to \infty$. Since A is invariant under g, it must then be the case that $A = 0$ on every leaf of $\mathcal{F}$. This implies that $a = 0$ on V, which is a contradiction. □

Lemma 38. *Suppose that U is a split region of $\widetilde{M}$ with at least one boundary component. If G acts on $\widetilde{M}$ by isometries fixed point freely and $G(U) \subset U$, then G is a free group.*

Proof. Recall from Theorem 31 that U is isometric to $\Sigma \times \mathbb{R}$ with Σ a subset of $\mathbb{H}^2$ with complete geodesics for its boundary components. Suppose for contradiction that there is a non-trivial $g \in G$ such that g fixes a point $p \in \Sigma$. Then let $r = \inf_{\gamma_j} d(p, \gamma_j)$ where $\{\gamma_j\}$ is the set of boundary components of Σ. There must be at least one boundary component that realizes the infimum and, moreover, only finitely many do, since the boundary components are complete geodesics in $\mathbb{H}^2$.

Then g must act on the set $\{\gamma_j | d(p, \gamma_j) = \min_i d(p, \gamma_i)\}$ of those boundary components. Since the set is finite, there is some $k > 0$ such that some γ_j is invariant under g^k. But then the boundary $\gamma_j \times \mathbb{R}$ of U is invariant under g^k. By the previous lemma, we know the only such isometries are trivial. Then g has order at most k, which contradicts the observation that G cannot have torsion without having a fixed point. So G acts fixed-point freely on Σ

Similarly, we can see that G acts properly discontinuously. Suppose that $p_0 \in U$ and there is a sequence of distinct points $p_i = g_i(p_0)$, $g_i \in G$ with $g_i \neq g_j$ with $p_i \to p_* \in U$. Then let γ_* be any geodesic of minimal distance to p_* and let $D = d(\gamma_*, p_*)$. For $\epsilon > 0$, choose N so that $d(p_i, p_*) < \epsilon$ for $i > N$. Each g_i has $g_i^{-1}(\gamma_*)$ a boundary geodesic and in particular since $d(p_i, p_*) < \epsilon$, $d(g_i^{-1}(\gamma_*), p_0) < D + \epsilon$. The set of boundary geodesics that are distance at most $D + \epsilon$ from p_0 is finite. Hence there exists $j > k > N$ such that there is a geodesic γ_0 of distance at most $D + \epsilon$ from p_0 so that $g_j(\gamma_0)$ and $g_k(\gamma_0)$ are both γ_*. Then $g_j^{-1} g_k$ must fix γ_0. This is a contradiction with the previous lemma. So G must act properly discontinuously as well as fixed-point freely.

Now Σ is an open surface that is contractible (since it is a convex subset of $\mathbb{H}^2$) and G acts on it fixed-point freely and properly discontinuously. So G is the fundamental group of Σ/G, a non-compact surface. Hence G is a free group, since a well-known result says that the fundamental group of any non-compact surface is free. See section 4.2.2 of [Sti93] for a reference. $\square$

Lemma 39. *Suppose that a complete manifold M has constant Ricci eigenvalue $(-1,-1,0)$ and is everywhere locally irreducible. Then $\pi_1(M)$ is either trivial or $\mathbb{Z}$.*

Proof. We use the strategy from the proof of Lemma 37. Specifically, pick a leaf P_0 of $\mathcal{F}$ of $\widetilde{M}$. Then define $A(P_x)$ by $\pm \int_0^x |a(\gamma(t))| \, \|\gamma'(t)\| \, dt$ where γ is any curve orthogonal to $\mathcal{F}$ with $\gamma(0)$ in P_0 and $\gamma(x)$ on P_x. (Choose the sign of $A(P_x)$ to be positive for $x > 0$ and negative for $x < 0$.) As shown in the previous lemma's proof, this is independent of our choice of γ.

This gives us a way to measure the distance between planes that is invariant under isometries. Specifically, $|A(P_x) - A(P_y)|$ defines a metric on the space of leaves of $\mathcal{F}$ of $\widetilde{M}$ and any isometry of $\widetilde{M}$ induces an isometry of this metric. The fact that $\widetilde{M}$ is nowhere reducible shows that $a(\gamma(t)) \neq 0$ on any interval and so A is strictly monotone and hence $A(P_x) = A(P_y)$ if and only if $P_x = P_y$. So A gives an identification of the space of leaves of $\mathcal{F}$ with $\mathbb{R}$ so that $\pi_1(M)$ acts on by isometries $\mathbb{R}$.

Lemma 37 shows that $\pi_1(M)$ acting on $\widetilde{M}$ cannot fix a leaf of $\mathcal{F}$. Hence $\pi_1(M)$ acts fixed point freely on $\mathbb{R}$. Moreover, we want to show that $\pi_1(M)$ acts properly discontinously on $\mathbb{R}$ and hence is either trivial or $\mathbb{Z}$. Suppose not. Then the orbit of any $x \in \mathbb{R}$ under $\pi_1(M)$ is dense in $\mathbb{R}$. Hence, if some leaf P has $a = 0$ on it, then $a = 0$ on all planes, by continuity of a and hence on all of M. This contradicts the assumption that M is locally irreducible. So $a \neq 0$ on M. We may now assume without loss of generality that $a > 0$ on M.

Hence we have a smooth foliation with coordinates (x, u, v) as in Proposition 18. Fix some $p_0 \in \widetilde{M}$ with $p_0 \in P_0$ and call $p_0 = (0,0,0)$. By assumption, there are $g_k \in \pi_1(M)$

such that $g_k(p_0)$ are in leaves P_k with $A(P_k) \to 0$ as $k \to \infty$. Since $\pi_1(M)$ acts properly discontinuously on $\widetilde{M}$, we must have that only finitely many of $p_k := g_k(p_0)$ are in any compact neighborhood of p_0. Let q_k be the point on each leaf P_k so that q_k lies on γ, so then $q_k \to p_0$ as $k \to \infty$. Letting $(x_k, u_k, v_k) = p_k$, if $u_k \to \pm\infty$ as $k \to \infty$ (on any subsequence), then $g_k^{-1}(q_k)$ must have u-coordinate $-u_k$ and lies in P_0. But since we know the form of $a(x, u, v) = f(x)/(\cosh u - h(x) \sinh u)$, either $a \to 0$ or $a \to \infty$ for $u \to \infty$ and x fixed. This is a contradiction since $a(0, -u_k, -v_k)$ must be the same as at $a(q_k)$ by the isometry g_k and $a(q_k) \to a(p_0)$ which is non-zero and finite.

Hence u_k must be bounded. Then v_k must diverge instead. As in [FZ16], we note that

$$T(e_2(a)) = e_2(T(a)) + [T, e_2](a) = (\nabla_T e_2 - \nabla_{e_2} T)(a) = ae_1(a) \tag{4.37}$$

and that $T(e_1(a)) = e_1(T(a)) + [T, e_1](a) = 0$. Hence $e_2(a) = ae_1(a)v + d$ for some d with d, a, and $e_1(a)$ all independent of v. Therefore $e_2(a)$ at $g_k^{-1}(q_k)$ must diverge as $k \to \infty$ since u is bounded and $x = 0$. But this means that $e_2(a)$ must diverge at p_0 since $q_k \to p_0$ and $e_2(a)$ is an isometry invariant up to sign. This gives a contradiction since $e_2(a)$ must be finite at any point. Hence $\pi_1(M)$ must actually act properly discontinuously on $\mathbb{R}$, and hence is trivial or $\mathbb{Z}$. $\square$

Now we prove the theorem.

Proof. Now consider the associated tree Γ_M of the universal cover$\widetilde{M}$. We have assumed that M is such that Γ_M is non-trivial, i.e. contains at least one "U" and one "V" vertex.

Since $\pi_1(M)$ acts as deck transformations on the cover $\widetilde{M} \to M$, $\pi_1(M)$ induces an action by graph isomorphisms on Γ_M of $\widetilde{M}$. The first and second lemmas say that the stabilizer group of any non-split region is trivial. Since every edge connects a non-split region to a split region, the stabilizer group of any edge is trivial as well (and in particular $\pi_1(M)$ cannot invert any edge). The third lemma says that the stabilizer group of any split region is a free group.

By Bass-Serre theory (see [Ser80], Theorem 13 from section 5.4), we can conclude that $\pi_1(M)$ is free as well. This theorem says that $\pi_1(M)$ is formed as the amalgamation of the stabilizers of some of the vertices of Γ_M amalgamated over the stabilizers of the edges. Since the stabilizers of our edges are trivial, the amalgamation reduces to a free product of stabilizers of vertices (which we have shown are free) along with a free group with generators corresponding to some edges. More specifically, $\Gamma_M/\pi_1(M)$ is a graph which has some spanning tree S. Then $\pi_1(M)$ is the free product of the stabilizers of any vertex in the graph (which are all free groups) and the free group generated by edges not in the spanning tree S. Hence $\pi_1(M)$ is free. □

Example 40. Any free group can be achieved as the fundamental group of some M with Ricci eigenvalues $(-1, -1, 0)$. To get this, recall that each free group is the fundamental group of some graph, namely the free group on k generators is $\pi_1(\Gamma)$ when Γ has k cycles. Then construct $\widetilde{M}$ so that $\Gamma_M = \Gamma$ so that there are isometries that make $\pi_1(M)$ the free group on k generators. Moreover, even a countably-generated free group is achievable, since the free group on two generators contains a countably-generated free group as a subgroup.

Example 41. There is a $\mathbb{Z}$ action on any metric of the form (4.13) with f, h smooth and periodic with the same period. Then the $\mathbb{Z}$ action is just by translation in x by the period of f and h.

Remark 42. Note that if we do not have any split points, then there are other possible fundamental groups. For example, $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{H}^2 \times \mathbb{R}$ with one $\mathbb{Z}$ acting on each factor.

Example 43. Note that the proof strategy in Theorem 33 shows that M simply connected deformation retracts to Γ_M. Moreover, for non-simply connected M we could associate a graph Γ_M, however the vertices of Γ_M would not correspond to the same geometric structures as in Theorem 33. In particular, regions may not be simply connected and so we cannot expect M to deformation retract to Γ_M in general. Example 40 shows that all free groups can be achieved as $\pi_1(M)$ for M that does deformation retract to the associated

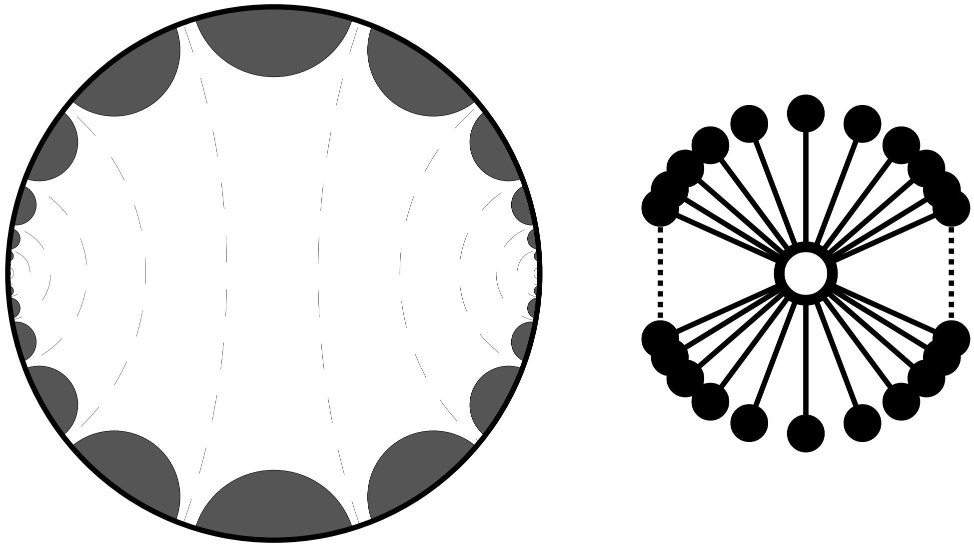

Figure 8: On the left is a diagram of $\widetilde{M}$ which is simply connected and has a $\mathbb{Z}$ action on it but $\widetilde{M}/\mathbb{Z}$ does not deformation retract onto $\Gamma_{\widetilde{M}}/\mathbb{Z}$. In particular $\widetilde{M}/\mathbb{Z}$ has a non-split region that is not simply connected.

graph and all regions are simply connected in M. However, Figure 8 is an example where a non-split region is not simply connected and M does not deformation retract onto its graph. In this figure, the dashed lines denote the fundamental domains of the $\mathbb{Z}$ action. This is an example where the stabilizer of a vertex of $\Gamma_{\widetilde{M}}$ is non-trivial. The graph $\Gamma_{\widetilde{M}}/\mathbb{Z}$ is a tree with three vertices but $\widetilde{M}/\mathbb{Z}$ is not simply connected.

4.6.1. Isometries

We end by computing the isometries of metrics of the form (4.13), as well as for other metrics that simplify the description of the isometries under the simplifying assumptions that $a > 0$ or that $a > 0$ and $|h| \neq 1$.

Proposition 44. *Suppose that M_i are manifolds $(a_i, b_i) \times \mathbb{R}^2$ with metrics given of the form in* (4.12) *defined by f_i, h_i with $|h_i| \leq 1$ and f_i non-zero on a dense set. Then any local isometry $\phi : M_1 \to M_2$ is of the form*

$$\phi(x,u,v) = (\ell^{-1}(\alpha x) + x_0, \beta u + u_0\cos(\theta(x)) + v_0\sin(\theta(x)), \alpha\beta v - u_0\sin(\theta(x)) + v_0\cos(\theta(x))) \tag{4.38}$$

for some fixed $x_0, r_0, \theta_0 \in \mathbb{R}$ *and* $\alpha, \beta \in \{\pm 1\}$ *where*

$$F(x) = \int_{x_0}^{x} f_2(s)ds,$$

$$\ell(t) = \int_{x_0}^{x_0+t} \cosh(u_0 \cos F(x) + v_0 \sin F(x)) - h_2(x) \sinh(u_0 \cos F(x) + v_0 \sin F(x))dx,$$

$$\theta(t) = F(\ell^{-1}(\alpha t) + x_0).$$

Moreover, such a mapping is an isometry if and only if

$$a_1(x, u, v) = a_2(\phi(x, u, v))$$

where

$$a_i(x, u, v) = \frac{f_i(x)}{\cosh(u) - h_i(x)\sinh(u)}.$$

Proof. Recall that $T = \frac{\partial}{\partial u}$ spans $\ker R$ and so $d\phi$ must take $T \in TM_1$ to $\pm T \in TM_2$. Similarly, on the set where $f_i(x) \neq 0$, $e_1 = \frac{\partial}{\partial v}$ spans $\ker C$. Extend e_1 smoothly by this definition to the entire domain. Since f is non-zero on an open dense set, we get that e_1 and a are isometry invariants, up to sign. So $d\phi$ must take $e_1 \in TM_1$ to $\pm e_1 \in TM_2$ everywhere. In particular, this means that the planes spanned by $\{T, e_1\}$ must be preserved under isometry. This then determines that $e_2 \in TM_1$ must be mapped to $\pm e_2 \in TM_2$. So choose α, β, δ such that $d\phi(e_2) = \alpha e_2$, $d\phi(e_1) = \beta e_1$, and $d\phi(T) = \delta T$.

Hence ϕ must take any e_2 integral curve in M_1 to an e_2 integral curve in M_2 with possibly opposite orientation. Recall that $\gamma(t) = (t, 0, 0) \in M_1$ is an e_2 integral curve. In general, e_2 is parallel to

$$X = \tfrac{\partial}{\partial x} + f(x)v\tfrac{\partial}{\partial u} - f(x)u\tfrac{\partial}{\partial v}.$$

The integral curves of X in M_2 are given by

$$\mu(t) = (t + x_0, u_0 \cos(F(t)) + v_0 \sin(F(t)), -u_0 \sin(F(t)) + v_0 \cos(F(t)))$$

for any constants $x_0, u_0, v_0 \in \mathbb{R}$ and $\alpha \in \{\pm 1\}$. So ϕ takes γ to μ for some choice of x_0, u_0, v_0, α. However, γ is arc-length parametrized but μ is in general not. Rather, the length of μ from $\mu(0)$ to $\mu(t)$ is given by ℓ defined above. So

$$\phi(\gamma(t)) = \mu(\ell^{-1}(t)).$$

Moreover, ϕ must take the planes of $\mathcal{F}$ at $\gamma(t)$ in M_1 to the planes of $\mathcal{F}$ at $\phi(\gamma(t))$ in M_2. Since ϕ preserves T, e_1 up to sign, ϕ restricted to any one of these planes is an isometry, and it can at most translate and reflect along the T and e_1 axes. Since we know that $\phi(\gamma(t)) = \mu(\ell^{-1}(t))$, we then know that the entire isometry is of the form

$$\phi(x, u, v) = (\ell^{-1}(x) + x_0, \beta u + u_0 \cos(\theta(x)) + v_0 \sin(\theta(x)), \alpha\beta v - u_0 \sin(\theta(x)) + v_0 \cos(\theta(x)))$$

for $\beta, \delta \in \{\pm 1\}$.

Next, we can check directly when ϕ is, in-fact, an isometry. We first compute $\frac{\partial \phi}{\partial x}$. This is given by

$$\begin{aligned}\tfrac{\partial \phi}{\partial x} = \alpha \frac{d\ell^{-1}}{dt}(\alpha x)\tfrac{\partial}{\partial x} &+ \left[-u_0 \sin(\theta(x)) + v_0 \cos(\theta(x))\right]\theta'(x)\tfrac{\partial}{\partial u} \\ &- \left[u_0 \cos(\theta(x)) + v_0 \sin(\theta(x))\right]\theta'(x)\tfrac{\partial}{\partial v}\end{aligned}$$

Observe that $\theta'(x) = \alpha f_2(\ell^{-1}(\alpha x) + x_0)\frac{d\ell^{-1}}{dt}(\alpha x)$ and

$$\begin{aligned}\frac{d\ell^{-1}}{dt}(\alpha x) = \big[&\cosh[u_0 \cos F(x_0 + \ell^{-1}(\alpha x)) + v_0 \sin F(x_0 + \ell^{-1}(\alpha x))] \\ &- h_2(x_0 + \ell^{-1}(\alpha x)) \sinh[u_0 \cos F(x_0 + \ell^{-1}(\alpha x)) + v_0 \sin F(x_0 + \ell^{-1}(\alpha x))]\big]^{-1}.\end{aligned}$$

Let $p = (x, u, v) \in M_1$ and $q = \phi(p) \in M_2$. Observe then that

$$\frac{\partial \phi}{\partial x} = \alpha \frac{d\ell^{-1}}{dt}(\alpha x)\left[\tfrac{\partial}{\partial x} + (v(q) - \delta v) f_2(x(q))\tfrac{\partial}{\partial u} - (u(q) - \beta u) f_2(x(q))\tfrac{\partial}{\partial v}\right] \tag{4.39}$$

and

$$\frac{d\ell^{-1}}{dx}(\alpha x) = [\cosh(u(q) - \beta u) - h_2(x(q))\sinh(u(q) - \beta u)]^{-1} \tag{4.40}$$

$$= \frac{f_1(x)}{f_2(x(q))} \tag{4.41}$$

since $a = f(x)/(\cosh u - h(x)\sinh u)$ is an isometry invariant and $u(q) - \beta u$ is $u(\phi(x, 0, 0))$, $x(q) = x(\phi(x, 0, 0))$, and $a_1(x, 0, 0) = f_1(x)$.

Using the metric g_2 and the form of $\frac{d\ell^{-1}}{dt}$ above,

$$\left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x} \right\rangle_{M_2} = \left[\frac{f_1(x)}{f_2(\ell^{-1}(\alpha x))}\right]^2 \Bigg[[\cosh(u(q)) - h_2(x(q))\sinh(u(q) - \beta u)]^2 \tag{4.42}$$

$$+ f_2(x(q))^2 \left[u^2 + v^2\right] \Bigg] \tag{4.43}$$

while in M_1, $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle_{M_1} = [\cosh(u) - h_1(x)\sinh(u)]^2 + f_1(x)^2(u^2 + v^2)$.

These two inner products are equal on the set where $a_i \neq 0$ if and only if

$$a_1(p)^{-2} + u^2 + v^2 = a_2(q)^{-2} + u^2 + v^2$$

which is equivalent to $a_1(p) = \pm a_2(\phi(q))$. Since $f_i^{-1}(x) = 0$ is discrete and a_i are smooth, we must have $a_1 = \pm\phi^* a_2$ everywhere.

Now we note that $\frac{\partial\phi}{\partial u} = \beta\frac{\partial}{\partial u}$ and $\frac{\partial\phi}{\partial v} = \delta\frac{\partial}{\partial v}$. With the above, we can then compute that all inner products are preserved when $a_1 = \phi^* \pm a_2$ plus the further condition that $\delta = \alpha\beta$ from computing $\left\langle \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial x} \right\rangle = -\alpha\beta\delta v f_1(x)$.

Hence, the condition $a_1 = \pm\phi^* a_2$ is necessary and sufficient for ϕ to be an isometry when ϕ has the form in the statement of the lemma. □

Under the simplifying assumption that $a > 0$, we can reparametrize a metric of the form (4.13) to the form (1.4), the original Sekigawa example. The isometries in these coordinates have a much simpler expression which have previously been studied [KTV90]. Remarkably, the isometries become translations in the x coordinate. This is due to the same fact that the function A defined in the proof of Lemma 37 (not related to our A in the following corollary) is independent of the choice of curve orthogonal to $\mathcal{F}$.

Corollary 45. *Suppose that M_i are manifolds $(a_i, b_i) \times \mathbb{R}$ with metrics given by*

$$g_i = (A_i(x) \cosh u + B_i(x) \sinh u)^2 dx^2 + (du - vdx)^2 + (dv + udx)^2$$

with $A_i(x) > 0$ and $|B_i(x)| \leq A_i(x)$. Then any isometry $\phi : M_1 \to M_2$ is of the form

$$\phi(x, u, v) = (\alpha x + x_0, \beta u + r_0 \sin(\alpha x + \theta_0), \alpha\beta v + r_0 \cos(\alpha x + \theta_0)) \tag{4.44}$$

for some fixed $x_0, r_0, \theta_0 \in \mathbb{R}$ and $\alpha, \beta \in \{\pm 1\}$. Moreover, such a mapping is an isometry if and only if the following conditions hold:

$$\begin{bmatrix} A_2(\alpha x + x_0) \\ B_2(\alpha x + x_0) \end{bmatrix} = \begin{bmatrix} \cosh(r_0 \sin(\alpha x + \theta_0)) & -\sinh(r_0 \sin(\alpha x + \theta_0)) \\ -\sinh(r_0 \sin(\alpha x + \theta_0)) & \cosh(r_0 \sin(\alpha x + \theta_0)) \end{bmatrix} \begin{bmatrix} A_1(x) \\ \beta B_1(x) \end{bmatrix} \tag{4.45}$$

Finally, the condition $|h(x)| < 1$ in metrics of the form (4.13) becomes $|B(x)| < |A(x)|$ in the Sekigawa-style coordinates above. This allows the term $A(x) \cosh u - B(x) \sinh u$ to be rewritten as $C(x) \cosh(u - u_0(x))$ for some smooth functions C, u_0, with $f > 0$. In this case, the isometries have an even simpler expression.

Corollary 46. *Suppose that M_i are manifolds $(a_i, b_i) \times \mathbb{R}$ with metrics given by*

$$g_i = C_i(x)^2 \cosh(u - u_i(x))^2 dx^2 + (du - vdx)^2 + (dv + udx)^2$$

with $C_i(x) > 0$ and $u_i(x)$ arbitrary. Then any isometry $\phi : M_1 \to M_2$ is of the form

$$\phi(x, u, v) = (\alpha x + x_0, \beta u + r_0 \sin(\alpha x + \theta_0), \alpha\beta v + r_0 \cos(\alpha x + \theta_0)) \tag{4.46}$$

for some fixed $x_0, r_0, \theta_0 \in \mathbb{R}$ *and* $\alpha, \beta \in \{\pm 1\}$*. Moreover, such a mapping is an isometry if and only if the following conditions hold:*

$$C_2(\alpha x + x_0) = C_1(x) \tag{4.47}$$

$$u_2(\alpha x + x_0) = \beta u_1(x) + r_0 \sin(\alpha x + \theta_0). \tag{4.48}$$

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