

ON CLOSED SIX-MANIFOLDS ADMITTING METRICS WITH
POSITIVE SECTIONAL CURVATURE AND NON-ABELIAN
SYMMETRY

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A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania in Partial
Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2019

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Acknowledgments

In the course of writing this thesis, I owe special thanks to my advisor, Wolfgang Ziller, for suggesting me the problem and bringing me inspiration whenever I struggled in doing my research. This thesis could not be done without his constant support and warm encouragement. I know that a few words cannot adequately express my thanks, I present this thesis as a token of my gratitude to him.

I would also like to thank Herman Gluck, Jonathan Block, Davi Maximo and Christopher Croke for providing all the support as well as being on my advisory committee or thesis committee. During my study at Penn, the mathematics that I had learned from them enriched my mathematical experience and will be the most valuable treasure of my life.

I also thank Fuquan Fang, Xiaochun Rong, Francisco Gozzi, Lee Kennard, Shi Wang and many other colleagues of mine whose expertise in this field gave me useful insights into the problem. I owe thanks to Mianlai Zhou, Artur Bicalho-Saturnino and Eric Goodman for helping me with the use of Mathematica and LaTeX graphics.

I want to share the credit of my thesis with many of my friends, those who

are always supportive and eager to offer help to me. I specially thank Zhen Zeng, Anusha Krishnan, Thomas Brooks, Joseph Hoisington and Jiaming Xia for making my life at Penn so enjoyable.

Finally I express my gratitude to my parents, who constantly comforted me when I was upset and encourage me to pursue my own career.

ABSTRACT

ON CLOSED SIX-MANIFOLDS ADMITTING METRICS WITH POSITIVE
SECTIONAL CURVATURE AND NON-ABELIAN SYMMETRY

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We study the topology of closed, simply-connected, 6-dimensional Riemannian manifolds of positive sectional curvature which admit isometric actions by $SU(2)$ or $SO(3)$. We show that their Euler characteristic agrees with that of the known examples, i.e. S^6 , \mathbb{CP}^3 , the Wallach space $SU(3)/T^2$ and the biquotient $SU(3)//T^2$. We also classify, up to equivariant diffeomorphism, certain actions without exceptional orbits and show that there are strong restrictions on the exceptional strata.

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Chapter 1

Introduction

The study of Riemannian manifolds with positive sectional curvature is an old and fundamental subject in Riemannian geometry. There are very few compact examples of positively curved manifolds besides the so-called Compact Rank One Symmetric Spaces, which we will abbreviate as CROSS. In fact the only further known examples exist only in dimension up to 24 and consist of homogeneous spaces [Wal72][BB76], biquotients [Esc82] and one cohomogeneity one manifold in dimension 7 [GVZ11][Dea11].

The fundamental group of a compact Riemannian manifolds with positive sectional curvature is finite, and it is trivial or equal to $\mathbb{Z}/2$ in even dimensions. Furthermore, odd-dimensional positively curved closed manifolds are orientable. However, for simply connected closed manifolds, no general topological obstructions are known to separate the class of positively curved manifolds from the class of non-

negatively curved manifolds, although there are many examples with non-negative curvature.

There are several classifications results of positively curved manifolds in low dimensions, though all of which require some “symmetry” conditions on the metric. Positively curved 3-manifolds are space forms [Ham82]. In dimension 4, Hsiang and Kleiner showed that positively curved simply connected 4-manifolds with S^1 symmetry are homeomorphic to the 4-sphere S^4 or the projective space \mathbb{CP}^2 (see [HK89]); later Grove and Wilking improved the result to equivariant diffeomorphism ([GW14]). In dimension 5, Xiaochun Rong showed that a T^2 -invariant simply connected closed 5-manifold is diffeomorphic to a 5-sphere (see [Ron02]).

Inspired by Hsiang and Kleiner’s work, Karsten Grove proposed what is now called the “symmetry program” in [Gro02], which is to study positively curved manifolds with “large” symmetry group. Here “large” can have several different meanings. Many results were obtained in this direction, particularly for torus actions. For example, Grove and Searle proved the Maximal Rank theorem ([GS94]), which states that the symmetry rank of a n -dimensional positively curved closed manifold is at most $\lceil \frac{n+1}{2} \rceil$, and in the case of equality the manifold is diffeomorphic to a sphere, \mathbb{RP}^n , \mathbb{CP}^n or a lens space. Burkhard Wilking showed that when $n \geq 10$ and a positively curved closed simply connected n -manifold M has symmetry rank at least $\frac{n}{4} + 1$, M is homeomorphic to S^n or $\mathbb{HP}^{\frac{n}{4}}$ or homotopy equivalent to $\mathbb{CP}^{\frac{n}{2}}$ [Wil03]. Recently Kennard, Wiemeler and Wilking showed that an even-

dimensional positively curved manifold with T^5 -symmetry has Euler characteristic at least 2.

After these classification results of positively curved manifolds invariant under torus actions, it is natural to investigate metrics with non-abelian symmetry. Wilking studied positively curved manifolds with high symmetry degree or low cohomogeneity relative to the dimension in [Wil06]. Since all non-abelian compact Lie groups contain a rank 1 subgroup, $SU(2)$ or $SO(3)$, it is natural to study metrics invariant under $SU(2)$ or $SO(3)$. In dimension 5, Fabio Simas obtained a partial classification of positively curved 5-manifolds invariant under $SU(2)$ or $SO(3)$ (see [Sim16]). In this paper we study 6-dimensional positively curved manifolds with $SU(2)$ or $SO(3)$ symmetry. This is also the first dimension where new examples have been constructed, which need to be recognized. They are the Wallach space $SU(3)/T^2$, where T^2 is the maximal torus, and the biquotient $SU(3)//T^2$, where $T^2 = \{(\text{diag}(z, w, zw), \text{diag}(1, 1, \bar{z}^2\bar{w}^2)) | z, w \in S^1\} \subset S(U(3) \times U(3))$ acts freely on $SU(3)$. On the first space one has an action by both $SO(3)$ and $SU(2)$, isometric in the positively curved metric, and on the second space an action by $SU(2)$ which commutes with $\text{diag}(1, 1, \bar{z}^2\bar{w}^2)$.

Our first result is:

Theorem 1.0.1. *Let $M = M^6$ be a 6-dimensional closed simply connected Riemannian manifold of positive sectional curvature such that $SU(2)$ or $SO(3)$ acts isometrically and effectively on M . Then:*

- (a) The Euler characteristic $\chi(M) = 2, 4, 6$;
- (b) The principal isotropy group is trivial unless M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$ -action;
- (c) When the principal isotropy is trivial, the exceptional isotropy groups are either cyclic or dihedral groups.

Notice that in the known examples, one has indeed $\chi(M) = 2, 4, 6$. Before stating the next theorems, we mention that the orbit space M/G is homeomorphic to a 3-sphere or a 3-ball (see Theorem 3.0.11) unless M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$ -action (See Chapter 6, Example 1(c)).

In the case of $G = SU(2)$ we will show:

Theorem 1.0.2. *Assume that $G = SU(2)$ acts on M isometrically and effectively.*

1. *If the fixed point set M^G is non-empty, then M is equivariantly diffeomorphic to a linear action on S^6 or \mathbb{CP}^3 .*
2. *If M^G is empty and the action has no exceptional orbits, then M is diffeomorphic to S^6 , $S^2 \times S^4$ or $SU(3)/T^2$.*

Explicit actions as in the above theorem are described in Chapter 6.

Theorem 1.0.3. *Suppose $G = SO(3)$ and assume that the orbit space M/G is a 3-ball whose boundary contains more than 1 orbit types, and that there are no exceptional orbits or interior singular orbits. Then M^6 is equivariantly homeomorphic to a linear action on S^6 .*

See Theorem 4.1.1 for further results in this special cases. The strategy to obtain these results is to analyze the structure of the orbit space and recover M from M/G . We will show that M/G is homeomorphic to B^4 , B^3 or S^3 . We describe the structure of singular orbit strata in all three cases, which allows us to glue different pieces of singular orbits to recover the topology of M if exceptional orbits do not occur. If exceptional orbits occur, we show that the stratification of M^* must be very special. We point out here that Fuquan Fang studied this problem, and claimed a partial classification of such positively curved 6-manifolds (see [Fan02]). Unfortunately, his proof contains several gaps.

Chapter 2

Preliminaries

We start by recalling some basic definitions for group actions, see e.g. [Bre72][AB15] for a reference. Let G be a compact Lie group and M be a compact smooth manifold. For a smooth action $\pi : G \times M \rightarrow M$, the *G-orbit* $G.p$ through a point $p \in M$ is the submanifold $G.p = \{gp \in M | g \in G\}$, the *isotropy group* or the stabilizer at $p \in M$ is defined as $G_p = \{g \in G | gp = p\}$, and we have $G.p = G/G_p$. Furthermore, we denote the *G-fixed point set* by $M^G = \{p \in M | G.p = p\}$. Note also that the fixed point set in an orbit has the form $(G/K)^H = \{g \in G | g^{-1}Hg \subset K\}/K$, where $H \subset K \subset G$. In particular, $(G/H)^H = N(H)/H$.

Points in the same G -orbits have conjugate isotropy groups. The *isotropy type* of a G -orbit G/H is the conjugacy class of isotropy groups at points in G/H and denote it by (H) . We define $M_{(K)}$ to be the union of orbits with the same isotropy type (K) . For compact group actions on compact manifolds, there are only finitely

many orbit types.

Among all orbit types of a given action, there exist maximal orbits G/H with respect to inclusion of isotropy groups called the *principal orbits*. Non-principal orbits which have the same dimension as the principal orbit are called *exceptional orbits*, and orbits having lower dimension than principal ones are called *singular orbits*.

The *orbit space* $M^* = M/G$ is the union of its *orbit strata* $M_{(K)}^* = M_{(K)}/G$ which themselves are manifolds. The principal orbit stratum $M_{(H)}^*$ is an open, dense and connected subset of M/G . In particular the dimension of $M_{(H)}^*$ is called the *cohomogeneity* of the action. Codimensional one strata in M^* are called *faces*, which are part of ∂M^* . We notice that $M_{(K)}^K \rightarrow M_{(K)}^*$ is an $N(K)/K$ -principal bundle and the structure group of $M_{(K)} \rightarrow M_{(K)}^*$ is $N(K)/K$.

The following theorem gives constraints on the exceptional orbits in simply-connected manifolds:

Theorem 2.0.4. ([Bre72]) *Let M be a simply-connected manifold and G a compact group acting on M . Then M^* is also simply connected and there are no exceptional orbits G/K whose stratum $M_{(K)}^*$ has codimension 1 in M^* (so called special exceptional orbits).*

For each orbit $G.p$, let T_p^\perp denote the normal space at p to the orbit and S_p^\perp the unit sphere in the normal space. T_p^\perp admits a natural linear action by the isotropy group G_p , called the *slice representation*. The quotient T_p^\perp/G_p is called the *tangent*

cone of the orbit in the orbit space, and S_p^\perp/G_p is the *space of directions* at p and is denoted as $S_{[p]}$. We also note that $M_{(K)} \cap T_p^\perp = (T_p^\perp)^K$, and the slice theorem states that an equivariant neighborhood of G/G_p has the form $G \times_{G_p} D(T_p^\perp) = (G \times D(T_p^\perp))/G_p$, where $D(T_p^\perp)$ is a disk in T_p^\perp (also called the *slice* at p) and G_p acts diagonally on G via right multiplication and on $D(T_p^\perp)$ via the slice representation.

If G acts on M by isometries, the orbit space, tangent cones and spaces of directions all inherit a metric from M . In particular, if we impose the positive curvature assumption on M , M/G becomes an Alexandrov space with positive curvature.

We frequently use the knowledge of the subgroups of $SO(3)$ and $SU(2)$. For $SO(3)$ they are given by

- 0-dimensional subgroups: \mathbb{Z}/k , D_k (dihedral groups acting on k vertices), A_4 , S_4 , A_5 ;
- 1-dimensional subgroups: $SO(2)$, $O(2)$;

and for $SU(2)$ by

- 0-dimensional subgroups: \mathbb{Z}/k , binary dihedral groups, inverse images of A_4 , S_4 , A_5 in $SU(2)$;
- 1-dimensional subgroups: $U(1)$, $Pin(2) = N(U(1))$.

Note that the only subgroups of $SU(2)$ which do not contain the center $\mathbb{Z}/2$ are cyclic groups of odd order.

It will also be useful for us to describe the quotient of \mathbb{R}^3 under a finite subgroup Γ of $SO(3)$. In the following pictures, a line segment represents a stratum of \mathbb{R}^3/Γ with indicated cyclic isotropy, the origin has isotropy Γ and the complement has trivial isotropy.

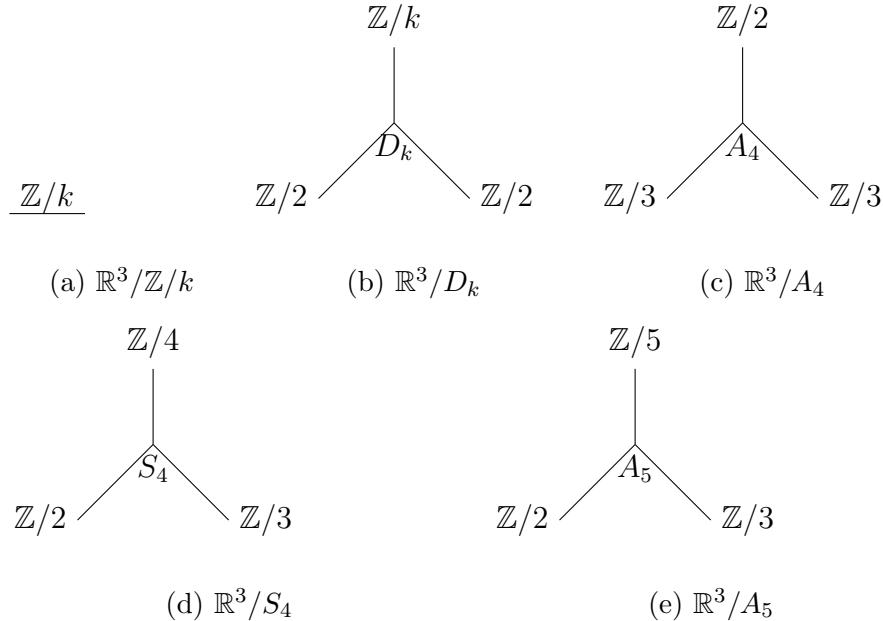


Figure 2.1: Finite quotients of \mathbb{R}^3

For the Euler Characteristic we have

Theorem 2.0.5. (a) ([Kob58]) If a torus T acts smoothly on a closed smooth manifold M , then the Euler characteristic of M equals that of M^T , that is,

$$\chi(M) = \chi(M^T);$$

(b) ([PS12]) If M is a 6-dimensional simply connected Riemannian manifold with positive sectional curvature and S^1 -symmetry, then $\chi(M)$ is positive and even.

For totally geodesic submanifolds of positively curved manifolds, we have the so called Connectedness Lemma due to Burkhard Wilking:

Theorem 2.0.6. (*Connectedness Lemma, [Wil03]*) Let M^n be a compact n -dimensional Riemannian manifold with positive sectional curvature. Suppose that $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k . Then the inclusion map $N^{n-k} \hookrightarrow M^n$ is $(n-2k+1)$ -connected.

Recall that if we have a continuous map $f : X \rightarrow Y$ between two connected topological spaces X and Y , and a positive integer k , then we say that f is k -connected if f_* induces isomorphisms on homotopy groups π_i , $1 \leq i \leq k-1$ and surjection on π_k .

For smooth actions on positively curved manifolds with nontrivial principal isotropy group, we have:

Theorem 2.0.7. (*Isotropy Lemma, [Wil06]*) Let G be a compact Lie group acting isometrically and not transitively on a positively curved manifold (M, g) with nontrivial principal isotropy group H . Then any nontrivial irreducible subrepresentation of the isotropy representation of G/H is equivalent to a subrepresentation of the isotropy representation of K/H , where K is an isotropy group such that the orbit stratum of K is a boundary face in M/G and K/H is a sphere.

Finally, for Riemannian manifolds with positive sectional curvature and low fixed point cohomogeneity, we have the following classification which will be used in the proof of Proposition 4.0.14.

Theorem 2.0.8. ([GS97]/[GK04]) If M is a positively curved simply connected closed manifold which admits an isometric action by a compact group G such that the fixed point cohomogeneity $\text{cohomfix}(M, G) := \dim(M/G) - \dim(M^G) - 1 \leq 1$, then M is equivariantly diffeomorphic to a compact rank one symmetric space.

We now state a version of the soul theorem in the setting of orbit spaces, which will be used in Theorem 4.1.1.

Theorem 2.0.9. (Theorem 1.2 [GK04], boundary soul lemma). Let M be a closed Riemannian manifold with positive sectional curvature and G a compact Lie group acting isometrically on M . Suppose $M^* = M/G$ has nonempty boundary ∂M^* . Then we have

1. There exists a unique point $s_o \in M^*$, the soul of M^* , at maximal distance to ∂M^* ;
2. The space of directions $S_{[s_o]}$ at s_o is homeomorphic to ∂M^* ;
3. The strata in $\text{int}(M^*) = M^* - \partial M^*$ belong to one of the following:
 - (a) all of $\text{int}(M^*)$;
 - (b) the soul point s_o ;
 - (c) a cone over strata in ∂M^* with its cone point s_o removed;
 - (d) a stratum containing s_o whose boundary consists of strata in ∂M^* .

Remark 2.0.10. We note that in [GK04] it was claimed that the strata in part (d) is one-dimensional, but one easily gives examples where its dimension is higher.

Chapter 3

The Structure of Orbit Spaces and Orbit Types

Throughout the remainder of the paper, G will always be the Lie group $SU(2)$ or $SO(3)$, and M is a simply connected closed 6-dimensional Riemannian manifold with positive sectional curvature which admits an effective isometric G -action.

We start by observing the following dichotomy for the topology of the orbit space M/G :

Theorem 3.0.11. *The orbit space M^* is homeomorphic to either S^3 or a 3-ball B^3 or B^4 . When $M^* = B^4$, M is equivariantly diffeomorphic to S^6 with a fixed point homogeneous linear $SO(3)$ -action.*

Proof. The cohomogeneity is calculated via $\dim(M^*) = \dim(M) - \dim(G) + \dim(H) = 3 + \dim(H)$, where H is the principal isotropy group. H is either 0 or 1-dimensional,

since closed subgroups of $G = SU(2)$, $SO(3)$ have dimensions 0,1,3, and H cannot be 3-dimensional since otherwise the G -action would be trivial. Thus the cohomogeneity is either 3 or 4.

Suppose that the cohomogeneity is 4. Then the principal isotropy group H is 1-dimensional, thus one of S^1 , $O(2)$ or $Pin(2)$. Since the isotropy representation of H is irreducible, Theorem 2.0.7 implies that the boundary face has isotropy $K = G$ and that the G -action is fixed point homogeneous. The only fixed point homogeneous action with cohomogeneity 4 is the linear $SO(3)$ -action on S^6 fixing the first 4 coordinates with quotient B^4 (Example 1(c) in Chapter 6).

When the cohomogeneity is 3, the orbit space M^* is a simply connected 3-dimensional topological manifold possibly with boundary, and [Bre72] Corollary 4.7 implies that M^* is homeomorphic to a 3-sphere with finitely many open disks removed. If ∂M^* is non-empty, the Soul Theorem implies that M^* is contractible. In conclusion, M^* is either a simply connected 3-manifold without boundary, thus a 3-sphere by Perelman's solution to the Poincare conjecture; or a 3-sphere with one open disk removed, thus a 3-ball. □

We note that we have 4 kinds of orbits, corresponding to the 0,1,3-dimensional closed subgroups of G :

- (a) Principal orbits G/H , with principal isotropy group H which will be shown to be trivial, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, or $SO(2)$;

- (b) Exceptional orbits G/Γ , with isotropy groups Γ which are finite extensions of H ; we will show Γ is cyclic or dihedral when H is trivial;
- (c) Singular orbits G/K , with 1-dimensional isotropy groups K , and hence $K = SO(2)$, $O(2)$ when $G = SO(3)$, and $K = U(1)$, $Pin(2)$ when $G = SU(2)$;
- (d) Fixed points i.e. $G_p = G$.

We now prove part (b) of Theorem 1.0.1, using Wilking's Isotropy Lemma.

Theorem 3.0.12. *The principal isotropy subgroup H is trivial unless M is equivariantly diffeomorphic to S^6 with a linear $SO(3)$ -action.*

Proof. Suppose the principal isotropy group H is non-trivial. By Theorem 3.0.11, we only need to consider the case $\dim(M^*) = 3$ and thus H is finite. If the isotropy representation of H on the tangent space to G/H has an irreducible subrepresentation of dimension greater than 1, then by the Isotropy Lemma, the isotropy K of the boundary face has dimension at least 2. Hence $K = G$, i.e. the G -action is fixed point homogeneous. In all other cases the irreducible components of the (3-dimensional) isotropy representation of G/H are 1-dimensional. Among the non-trivial subgroups of G , only $\mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in $SO(3)$ has 3-dimensional representations of this type. Thus $H = \mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and $G = SO(3)$. Since the isotropy representation of $SO(3)/H$ has a 1-dimensional subrepresentation on which H acts as -Id, the Isotropy Lemma implies that the boundary face has isotropy $K = O(2)$. The only higher strata on ∂M^* are fixed points. We note here that Proposition

4.0.14 does not depend on the triviality of H , thus it is valid to quote its proof.

From the proof of Proposition 4.0.14, if $M^G \neq \emptyset$ and the principal isotropy is non-trivial, then the G -action on M has fixed point cohomogeneity at most 1 and M is equivariantly diffeomorphic to S^6 with a linear action.

So we can assume that $M^G = \emptyset$, the boundary of the orbit space has isotropy $O(2)$ and hence the interior regular part has isotropy $H = \mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. From Theorem 2.0.9(3), we have two possibilities: either M^* has one interior singular orbit, or it has none. We rule out both possibilities by calculating the fundamental group and cohomology groups of M . From now on, $\pi : M \rightarrow M^*$ is the natural projection, $U = \pi^{-1}(\text{int}(M^*))$ and V is a tubular neighborhood of $\pi^{-1}(\partial M^*)$. $M = U \cup V$ is the desired decomposition.

1. If M^* has no interior singular orbit, then U is an $SO(3)/H$ -bundle over $\text{int}(M^*) = D^3$ and thus $U = SO(3)/H \times D^3$ since the base D^3 is contractible. V deformation retracts onto an $SO(3)/O(2)$ -bundle over $\partial M^* = S^2$, with structure group $N(O(2))/O(2) = id$. Thus V retracts onto a trivial \mathbb{RP}^2 -bundle over S^2 . $U \cap V$ deformation retracts onto $SO(3)/H \times S^2$. Let $i : U \cap V \rightarrow U$, $j : U \cap V \rightarrow V$ denote the respective inclusions. By van Kampen's theorem, $\pi_1(M) = \pi_1(U) * \pi_1(V) / \langle i_*(a)j_*(a)^{-1} | a \in \pi_1(U \cap V) \rangle$. It is non-trivial since i_* is an isomorphism and thus $\pi_1(V) \cong \mathbb{Z}/2$ cannot be killed. This contradicts the assumption that M is simply connected.

2. If M^* has an interior singular orbit, a priori the singular orbit could have

isotropy $SO(2)$ or $O(2)$. Take the $SO(2)$ -fixed point set $M^{SO(2)} = V^{SO(2)} \cup U^{SO(2)}$. Since $SO(2)$ fixes one point in each orbit on ∂M^* , $V^{SO(2)}$ is homeomorphic to $\partial M^* = S^2$. If the interior singular orbit has isotropy $O(2)$, then $U^{SO(2)} = (SO(3)/O(2))^{SO(2)} = \text{pt}$. So Theorem 2.0.5 implies that $\chi(M) = \chi(M^{SO(2)}) = \chi(S^2 \cup \text{pt}) = 3$, contradicting the fact that $\chi(M)$ is even. Thus the interior singular orbit has isotropy $SO(2)$. Then the slice representation of $SO(2)$ has slope $(2,2)$, $H = \mathbb{Z}/2$ and $U^{SO(2)} = (S^2)^{SO(2)} = 2$ points. Therefore $\chi(M) = \chi(M^{SO(2)}) = \chi(S^2 \cup \{2 \text{ points}\}) = 4$. We first show that M has the cohomology groups of \mathbb{CP}^3 , and then use the Mayer-Vietoris sequence to obtain a contradiction.

M^H is a totally geodesic submanifold with even codimension since the H -action on M is orientation-preserving, and it has dimension at least 3 since H fixes at least one point in each principal orbit. Thus M^H is 4-dimensional submanifold with positive sectional curvature. Wilking's connectedness lemma implies that the inclusion of $i : M^H \hookrightarrow M$ is 3-connected. In particular, $\pi_1(M^H) = \pi_1(M) = 0$. Moreover, $N(H)/H = O(2)$ acts effectively on M^H . By the Hsiang-Kleiner theorem ([HK89]), M^H is diffeomorphic to S^4 or \mathbb{CP}^2 . If $M^H = S^4$, then Wilking's connectedness lemma implies M is a homology 6-sphere, violating $\chi(M) = 4$. Thus $M^H = \mathbb{CP}^2$. We then have $\pi_2(M) = \pi_2(\mathbb{CP}^2) = \mathbb{Z}$, $\pi_3(M) = i_*(\pi_3(\mathbb{CP}^2)) = 0$. Hence the Hurwitz theorem and Poincare duality imply that M has the cohomology groups of \mathbb{CP}^3 .

Now we apply the Mayer-Vietoris sequence to recompute $H^*(M)$. From the slice theorem, $U = SO(3) \times_{SO(2)} D^4$ and the slope of the $SO(2)$ action on D^4 is (2,2). U deformation retracts onto S^2 . V is an $SO(3)/O(2)$ -bundle over S^2 with structure group $N(O(2))/O(2) = \text{id}$ and thus $V = S^2 \times \mathbb{RP}^2$. $U \cap V = SO(3) \times_{SO(2)} S^3 = S^2 \times S^3$ since the sum of the slopes is even (see [Sim16]). We then have the following short exact sequence:

$$\begin{aligned} 0 \rightarrow H^2(M) &\cong \mathbb{Z} \rightarrow H^2(U) \oplus H^2(V) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow H^2(U \cap V) \cong \mathbb{Z} \rightarrow \\ &\rightarrow H^3(M) = 0, \end{aligned} \tag{3.0.1}$$

which leads to a contradiction.

Thus in both cases H is trivial. \square

In our setting, exceptional orbits could have rich and complicated structure, as we will see in the next chapters, making it difficult to recover the original manifold from the orbit space. This is also an issue ignored in Fang's paper [Fan02]. We state and prove some results on the structure of exceptional orbits, including Theorem 1.0.1(c).

Proposition 3.0.13. *For exceptional orbits G/Γ , the following holds:*

- (a) *The exceptional isotropy groups Γ are cyclic of odd order if $G = SU(2)$, and cyclic or dihedral if $G = SO(3)$;*

(b) In the orbit space M/G , there are no isolated exceptional orbit strata, that is, no exceptional strata whose closure does not contain singular strata.

Proof. We prove part (a) via case-by-case analysis. The finite subgroups of $SO(3)$

are \mathbb{Z}/m , D_n , A_4 , S_4 , A_5 .

If $\Gamma = A_4$, then the local picture of the exceptional strata is Figure 2.1c. We take the $\mathbb{Z}/2$ -fixed point set, and note that

$$\begin{aligned} (SO(3)/A_4)^{\mathbb{Z}/2} &= \{g \in SO(3) | g^{-1}(\mathbb{Z}/2)g \subset A_4\}/A_4 = N(\mathbb{Z}/2)A_4/A_4 \\ &= N(\mathbb{Z}/2)/N(\mathbb{Z}/2) \cap A_4 = N(\mathbb{Z}/2)/(\mathbb{Z}/2) = S^1 \coprod S^1, \\ (SO(3)/(\mathbb{Z}/2))^{\mathbb{Z}/2} &= N(\mathbb{Z}/2)/(\mathbb{Z}/2) = S^1 \coprod S^1, \quad (SO(3)/(\mathbb{Z}/3))^{\mathbb{Z}/2} = \emptyset. \end{aligned}$$

We conclude that $M^{\mathbb{Z}/2}$ has two circle boundaries at $SO(3)/A_4$. On the other hand, each component of $M^{\mathbb{Z}/2}$ is a 2-sphere, which is a contradiction.

If $\Gamma = S_4$, then the local picture of the exceptional strata is Figure 2.1d. Considering $M^{\mathbb{Z}/3}$ we have

$$\begin{aligned} (SO(3)/S_4)^{\mathbb{Z}/3} &= N(\mathbb{Z}/3)/(\mathbb{Z}/3) = S^1 \coprod S^1, \quad (SO(3)/(\mathbb{Z}_3))^{\mathbb{Z}/3} = N(\mathbb{Z}/3)/(\mathbb{Z}/3) \\ &= S^1 \coprod S^1, \\ (SO(3)/(\mathbb{Z}_2))^{\mathbb{Z}/3} &= \emptyset, \quad (SO(3)/(\mathbb{Z}_4))^{\mathbb{Z}/3} = \emptyset. \end{aligned}$$

Hence $M^{\mathbb{Z}/3}$ has circle boundaries at G/Γ , a contradiction.

If $\Gamma = A_5$, then the local picture of the exceptional strata is Figure 2.1e. Con-

sidering $M^{\mathbb{Z}/5}$ we have

$$\begin{aligned} (SO(3)/A_5)^{\mathbb{Z}/5} &= N(\mathbb{Z}/5)/(\mathbb{Z}/5) = S^1 \coprod S^1, \quad (SO(3)/(\mathbb{Z}_5))^{\mathbb{Z}/5} = N(\mathbb{Z}/5)/(\mathbb{Z}/5) \\ &= S^1 \coprod S^1, \\ (SO(3)/(\mathbb{Z}_2))^{\mathbb{Z}/5} &= \emptyset, \quad (SO(3)/(\mathbb{Z}_3))^{\mathbb{Z}/5} = \emptyset. \end{aligned}$$

Thus $M^{\mathbb{Z}/5}$ again has circle boundaries at G/Γ , a contradiction.

In conclusion, $\Gamma \neq A_4, S_4, A_5$, and hence the exceptional isotropy groups are cyclic or dihedral.

To prove part (b), we first observe that exceptional orbit strata cannot be 2-dim, as there are no special exceptional orbits. Thus they are isolated points or 1-dim curves. We want to show that there are no connected components of exceptional strata whose closure does not contain singular orbits, in particular, exceptional points can not be isolated.

Suppose there is a component of exceptional strata which is closed. Then it is a connected graph, which is a union of circles and intervals. Take some exceptional isotropy group K of the strata and a non-trivial cyclic subgroup C of K , and consider the C -fixed point component in this exceptional stratum.

In each exceptional orbit G/K , $(G/K)^C$ is the union of several circles. Since the exceptional strata are 1-dim, the fixed point component of M^C is 2-dim, and hence is a 2-sphere, as it is orientable, totally geodesic and hence has positive curvature. This induces a foliation of S^2 by circles, which is impossible since the tangent bundle of S^2 does not contain any sub line bundle. \square

Chapter 4

Actions, Orbit Spaces and the Topology of G-manifolds

In this chapter we study different types of G-actions on positively curved 6-manifolds.

We start with the case of a non-empty fixed point set.

Proposition 4.0.14. *If $M^G \neq \emptyset$, then one of the following holds:*

1. *M is equivariantly diffeomorphic to S^6 or \mathbb{CP}^3 with a linear action;*
2. *$G = SO(3)$ and M^G is finite. In this case $M^* = B^3$ and M^G lies on ∂M^* .*

Proof. We separate the cases of $SU(2)$ and $SO(3)$ actions.

- Case 1: $G = SU(2)$.

$SU(2)$ acts on the normal space to M^G effectively without fixed points.

By considering the faithful real representations without trivial summands of

$SU(2)$ in dimensions less than 6, we see that only the 4-dimensional irreducible representation of $SU(2)$ satisfies the requirements. Thus the codimension of M^G is 4 and the action of G on the normal space is equivalent to the realification of the standard $SU(2)$ -action on \mathbb{C}^2 . Hence G acts transitively on the unit sphere in \mathbb{C}^2 , the G action on M is fixed point homogeneous and Theorem 2.0.8 implies the desired result.

- Case 2: $G = SO(3)$.

For the action of $SO(3)$ on the normal space to M^G we have the following possibilities:

- (a) \mathbb{R}^3 with the standard $SO(3)$ -action. In this case the action is fixed point homogeneous;
- (b) \mathbb{R}^5 with the unique 5 dim irreducible representation of $SO(3)$. The $SO(3)$ -action on the unit normal sphere S^4 has cohomogeneity one, which by definition implies the G -action on M has fixed point cohomogeneity one;
- (c) $\mathbb{R}^3 \oplus \mathbb{R}^3$ with diagonal action of $SO(3)$. Note: Example 1(e) in Chapter 6 comes from a suspension of this isotropy representation. Thus the origin is an isolated fixed point.

If case (a) or (b) occurs, the G -action on M is equivariantly diffeomorphic to a linear action on S^6 or \mathbb{CP}^3 by Theorem 2.0.8. In case (c), the fixed points

are isolated and hence M^G is finite.

Finally it remains to show $M^* = B^3$. First note that from the above discussion, the isotropy representation of $SO(3)$ at an isolated fixed point is $\mathbb{R}^3 \oplus \mathbb{R}^3$ with diagonal action. The orbit types of this representation are:

- principal orbits with trivial isotropy, represented by two linearly independent vectors in \mathbb{R}^3 ;
- singular orbits with $SO(2)$ -isotropy, represented by two linearly dependent vectors in \mathbb{R}^3 which are not both zero;
- the fixed point $(0, 0)$.

The union of singular orbits near a fixed point has dimension 4 in M , which descends to 2-dimensional strata of M^* . Since $\dim(M^*) = 3$, this strata is a boundary face in ∂M^* and hence $M^* = B^3$ by Theorem 3.0.11.

□

Comparing with the class of linear actions in Chapter 6, one sees that in Case 1 the only actions are those given in Examples 1(b) and 2(a), while for Case 2(a) and 2(b) the actions are given by Examples 1(c) and 1(d) respectively. Next, we study the case where M^G is finite or empty.

4.1 $SO(3)$ actions with M^G finite or empty

By Theorem 3.0.11, we divide this section into two parts, corresponding to $M^* = B^3$ and $M^* = S^3$, and we start with the case where $M^* = B^3$.

Theorem 4.1.1. *Assume $G = SO(3)$, with $M^* = B^3$ and M^G finite. Then:*

1. *The boundary faces of M^* consists of singular orbits with $SO(2)$ -isotropy.*
2. *There is at most 1 interior singular orbit whose isotropy group has to be $SO(2)$.*
3. *If ∂M^* contains more than 1 orbit types, then ∂M^* contains exactly 2 singular points which are either two fixed points or one fixed point and one $O(2)$ -orbit. Moreover, if there is an $O(2)$ -orbit on ∂M^* , then there exist an interior singular orbit and a $\mathbb{Z}/2$ -exceptional stratum connecting the interior singular orbit and the $O(2)$ -orbit.*
4. *The Euler characteristic $\chi(M) \leq 6$. If ∂M^* contains more than 1 orbit types, then $\chi(M) \leq 4$.*

For part 3, see Example 2(c) in Chapter 6.

Proof. Proof of part 1: First of all by Theorem 2.0.4 the boundary does not contain any exceptional orbits. A priori the boundary face orbits could be singular orbits with $O(2)$ -isotropy, but then from the slice action of $O(2)$ on the 4-dimensional

normal space the principal isotropy group would be non-trivial (containing \mathbb{Z}_2), contradicting Theorem 1.0.1(b).

Proof of part 3: In each boundary face orbit, $SO(2)$ fixes exactly 2 points, since a boundary face orbit has $SO(2)$ -isotropy; in $O(2)$ -orbits on $\partial(M/G)$ or G fixed points, $SO(2)$ fixes one point. Thus the $SO(2)$ -fixed point component over $\partial(M/G)$ is a branched double cover of $\partial(M/G) = S^2$ with branching points corresponding to $O(2)$ -orbits or G -fixed points. Moreover, the $SO(2)$ -fixed point component is a 2-sphere itself, as it is orientable and has positive curvature. From the Riemann-Hurwitz formula, a branched double cover between two 2-spheres has exactly 2 branched points.

If there is an $O(2)$ -orbit on ∂M^* , then $O(2)$ acts on the 4-dim slice as $\text{diag}(1, 1, 1, -1)$. The $O(2)$ -action on the last two coordinates is effective since otherwise the principal isotropy would be non-trivial. So we get a $\mathbb{Z}/2$ -stratum emanating from the $O(2)$ -orbit which must end at the interior singular orbit. Finally there cannot be two $O(2)$ -orbits since otherwise the slice representation of the interior singular orbit would have slope $(2,2)$, forcing the principal isotropy group to be non-trivial.

Proof of part 2: Theorem 2.0.9(3) implies that M^* has at most 1 interior singular orbit G/K . A priori K could be $SO(2)$ or $O(2)$. Suppose $K = O(2)$. The slice representation of $O(2)$ is 4-dimensional and orientation-reversing, since the isotropy representation of $O(2)$ on $SO(3)/O(2)$ is orientation-reversing. We list all possible effective orientation-reversing $O(2)$ -actions on \mathbb{R}^4 :

Suppose the $SO(2)$ subgroup acts as $R(\theta) \mapsto \begin{bmatrix} R(p\theta) & 0 \\ 0 & R(q\theta) \end{bmatrix}$, where p and q are coprime integers since otherwise H is nontrivial. Let $\tau \in O(2) \setminus SO(2)$ be a reflection.

1. $p, q \neq 0$, and τ acts by $\text{diag}(1, -1, 1, -1)$. In this case the action of τ is orientation preserving, which implies that the slice action of $O(2)$ is orientation preserving. But this is not allowed.
2. $p = 0, q = 1$, and τ acts by $\text{diag}(1, 1, 1, -1)$. In this case the strata $M_{(O(2))}^*$ is 2-dimensional and thus $M_{(O(2))}^* \subset \partial M^*$, contradicting the assumption $G/K \in \text{int}(M^*)$.
3. $p = 0, q = 1$, and τ acts by $\text{diag}(-1, -1, 1, -1)$. In this case $M_{(SO(2))}^*$ is 2-dimensional and $M_{(SO(2))}^* \subset \partial M^*$. G/K lies in the closure of $M_{(SO(2))}^*$, and thus $G/K \in \partial M^*$. Thus this also cannot occur.

In conclusion $K = SO(2)$.

Proof of part 4: If ∂M^* has more than 1 orbit types, then there are two singular points in ∂M^* by part 3 and at most one singular orbit in the interior which has $SO(2)$ -isotropy. These include all singular orbits, and $M^{SO(2)}$ is either a 2-sphere or the union of 2-sphere with 2 points. Thus $\chi(M) = \chi(M^{SO(2)}) \leq 4$. If ∂M^* has only 1 orbit type, then Theorem 2.0.9(3) implies that either $\text{int}(M^*)$ is a stratum, or $\text{int}(M^*) - s_0$ and the soul point are two strata. In other words, M^* has no

exceptional orbits. Part 2 implies that the soul point has $SO(2)$ -isotropy. Thus $M^{SO(2)} = S^2 \cup S^2$ or $S^2 \cup S^2 \cup \{2 \text{ points}\}$ and $\chi(M) = \chi(M^{SO(2)}) = 4, 6$. \square

Corollary 4.1.2. *If $G = SO(3)$ and $M^* = B^3$, then the structure of M^* is as in Figures 4.1a, 4.1b below or the Figures 5.1a, 5.1b, 5.4a and 5.4b in Chapter 5. In the pictures, the groups represent the isotropy groups of the corresponding strata.*

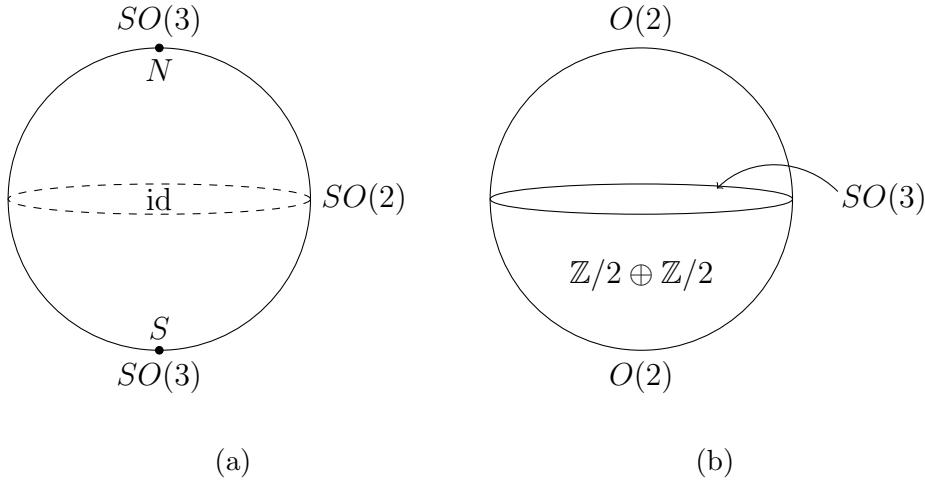


Figure 4.1: $G = SO(3)$, $M^* = B^3$

Proof. If M^G is not finite, Proposition 4.0.14 implies that M^* is as in Figure 4.1b. So we may assume that M^G is finite. Then Theorem 4.1.1 implies that the boundary face has isotropy $SO(2)$ and that there is at most 1 interior singular orbit.

Assume that ∂M^* has 1 orbit type. If $\text{int}(M^*)$ contains no singular orbit, M^* is as in Figure 5.1a. If $\text{int}(M^*)$ contains 1 singular orbit, M^* is as in Figure 5.4a.

Assume that ∂M^* has multiple orbit types. Theorem 4.1.1 implies ∂M^* contains two fixed points, or one fixed point and one $O(2)$ -orbit. If ∂M^* contains two fixed

points, M^* is as in Figure 4.1a or Figure 5.4b. Otherwise, M^* is as in Figure 5.1b. \square

We point out that in Figure 4.1a and 4.1b, M is classified. See Theorem 1.0.3 and Proposition 4.0.14 respectively. We now prove Theorem 1.0.3.

Theorem 4.1.3. *Assume that $G = SO(3)$, $M^* = B^3$, ∂M^* contains more than 1 orbit types and that there are no exceptional orbits or interior singular orbits. Then: M^6 is equivariantly homeomorphic to a 6-sphere S^6 and $G = SO(3)$ acts on $S^6 \subset \mathbb{R}^7$ linearly as in Example 1(e).*

Proof. From Theorem 4.1.1(3), we know ∂M^* has 2 singular points. Since M has no exceptional orbits, these two orbits cannot be $O(2)$ -orbits, otherwise there will be exceptional orbits near the $O(2)$ orbit with isotropy containing $\mathbb{Z}/2$. Thus the two singular points are two G -fixed points. Now we see that all orbit types are: principal orbits with trivial isotropy, singular orbits on ∂M^* with $SO(2)$ -isotropy and two G -fixed points on ∂M^* . We have assumed that there are no interior singular orbits. We need to classify G -spaces with 3 orbit types $(H) = (id)$, $(K) = (SO(2))$, $(G) = (SO(3))$ such that the number of fixed points is 2.

We recall the Second Classification Theorem in [Bre72]. For a smooth G -action on M , suppose the orbit space $X = M^*$ is a contractible manifold with boundary B , and that the action has only two orbit types, with principal orbits G/H corresponding to $X \setminus B$ and singular orbits G/K corresponding to B . Then the set of

equivalence classes of such G-spaces M is parametrized by the following set

$$[B, (N(H) \cap N(K)) \setminus N(H)]/\pi_0\left(\frac{N(H)}{H}\right)$$

where $[X, Y]$ denotes the homotopy classes of continuous maps from X to Y. (See Corollary V.6.2, page 257 of [Bre72])

For actions with 3 orbit types H, K and G, Proposition V.10.1 [Bre72] states that the set of equivariant homeomorphism classes of G-spaces with 3 orbit types is bijective to the set of equivariant homeomorphism classes of G-spaces with 2 orbit types (H) and (K) obtained by deleting the fixed points. The latter G-spaces are homotopy equivalent to G-spaces with orbit space a two-disk D^2 and 2 orbit types $H = id$, $K = SO(2)$ where the singular orbits G/K lie on the boundary of D^2 .

Those G-spaces are classified by

$$[\partial(D^2), (N(H) \cap N(K)) \setminus N(H)]/\pi_0\left(\frac{N(H)}{H}\right) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2.$$

Actually we can write down explicitly the 2 G-spaces. They are:

- the 5-sphere where the G-action comes from the restriction of the 6-dimensional real representation $\mathbb{R}^3 \oplus \mathbb{R}^3$ and $G = SO(3)$ acts diagonally;
- $S^2 \times S^3$ where $G = SO(3)$ acts diagonally on S^2 -factor as the standard linear action and on S^3 -factor as the linear suspension.

M is the suspension of the above 5-manifolds. But M is a manifold, so it can only be the suspension of the 5-sphere, which a 6-sphere, and the action is the one described in the theorem. \square

Proposition 4.1.4. *If $G = SO(3)$ and $M^* = S^3$, then there are 2 or 3 singular orbits.*

We first state the Extent Lemma. For any metric space (X, d) and positive integer $q \geq 2$, we define the q -extent of X as

$$xt_q(X) = \frac{1}{\binom{q}{2}} \sup_{x_1, \dots, x_q \in X} \sum_{1 \leq i < j \leq q} d(x_i, x_j). \quad (4.1.1)$$

In other words, $xt_q(X)$ is the maximal average distance between points in q -tuples in X . When $q=2$, $xt_2(X)$ is the diameter of X . The Extent Lemma from [GS97] states that: if M/G is an Alexandrov space with positive curvature, then for all $(q+1)$ -tuples $([x_0], \dots, [x_q])$ in M/G , we have:

$$\frac{1}{q+1} \sum_{i=0}^q xt_q(S_{[x_i]}) > \frac{\pi}{3}.$$

We can use the Extent Lemma to prove Proposition 4.1.4.

Proof. There exist singular orbits since $\chi(M^{S^1}) = \chi(M) > 0$ and thus $M^{S^1} \neq \emptyset$.

We apply the Extent Lemma to show that there are at most 3 singular orbits.

Suppose there were 4 singular orbits. Then in M^* , each singular orbit G/K_i has a space of directions $S^3(1)/K_i$, where $K_i = S^1 \text{or} O(2)$ acts linearly on the unit normal sphere $S^3(1)$ via the slice representation. $xt_3(S^3/K_i) \leq xt_3(S^2(\frac{1}{2})) = \frac{\pi}{3}$. Thus $\frac{1}{4} \sum_{i=0}^3 xt_3(S^3(1)/K_i) \leq \frac{\pi}{3}$, and we get a contradiction to the Extent Lemma.

Finally we show that there cannot be only 1 singular orbit. Suppose there were only 1 singular orbit G/K . Then there are no exceptional orbits and the singular

isotropy $K = S^1$. By the slice theorem, an invariant neighborhood of G/K is $SO(3) \times_{S^1} D^4$ whose boundary is an S^3 -bundle over S^2 . On the other hand the compliment of $SO(3) \times_{S^1} D^4$ in M is $SO(3) \times D^3$ whose boundary is $SO(3) \times S^2$. The two boundaries are not homeomorphic since their fundamental groups are not isomorphic, and thus there is no way of gluing the two pieces to get M . \square

4.2 $SU(2)$ actions with $M^G = \emptyset$

Proposition 4.2.1. *When $G = SU(2)$ and $M^G = \emptyset$, the orbit space M^* is a 3-sphere. Moreover, the fixed point set $M^{\mathbb{Z}/2}$ of the center $\mathbb{Z}/2$ is precisely the union of all singular orbits, which are all 2-spheres. Furthermore, there can be at most 3 singular orbits.*

Proof. We prove the second part first. Each component of $M^{\mathbb{Z}/2}$ is a totally geodesic orientable submanifold of even codimension in M and $SO(3) = SU(2)/(\mathbb{Z}/2)$ acts on it. A priori it could have dimension 4,2,0. But 0-dimensional components would be G -fixed points, violating our assumption. We then show that it cannot have dimension 4.

If a component of $M^{\mathbb{Z}/2}$ has dim 4, then the induced metric has positive sectional curvature and is invariant under $SO(3)$. From Wilking's connectedness lemma, it is also simply connected. Thus by the Hsiang-Kleiner theorem it is diffeomorphic to either S^4 or \mathbb{CP}^2 . And it also admits a cohomogeneity one action by $SO(3)$. From the classification of 4-dim cohomogeneity one manifolds (see for example [Par86]),

such actions have at least one singular orbit with $O(2)$ -isotropy, which lifts up to $Pin(2)$ -isotropy for the corresponding $SU(2)$ -action. The action of the $Pin(2)$ -isotropy group on the normal space to $M^{\mathbb{Z}/2}$ has to be effective, since otherwise the center $\mathbb{Z}/2$ would lie in the ineffective kernel. But this is impossible since the normal space is 2-dimensional and $Pin(2)$ has no effective 2-dim real representation. Thus every component of $M^{\mathbb{Z}/2}$ is a 2-dimensional orientable positively curved manifold, which is a 2-sphere. Those 2-spheres are precisely the singular orbits, since every $SO(2) \subset SU(2)$ contains $\mathbb{Z}/2$ and hence every singular orbit is contained in $M^{\mathbb{Z}/2}$.

To show that there are at most 3 singular orbits, we use the Extent Lemma. If there are 4 of them then as in the proof of Proposition 4.1.4, we get a contradiction to Extent Lemma.

Finally it remains to show $M^* = S^3$. Assume otherwise. Then by Theorem 3.0.11 $M^* = B^3$. The boundary of M^* consists of singular orbits, which means $M^{\mathbb{Z}/2}$ is 4-dimensional since it contains all singular orbits, which is impossible.

□

Remark 4.2.2. The above proposition says more than the statement that the orbit space has no boundary. In fact, there are also no exceptional orbits whose isotropy groups contain the center $\mathbb{Z}/2$, as a corollary. Hence the exceptional isotropy groups are all cyclic of odd order.

Now we prove Theorem 1.0.2.

Theorem 4.2.3. *$G = SU(2)$. If M^G is empty and the action has no exceptional orbits, then M is diffeomorphic to S^6 , $S^2 \times S^4$ or $SU(3)/T^2$.*

Proof. From Proposition 4.2.1, we know that $M^* = S^3$ and there are at most 3 singular orbits, all of which have $U(1)$ -isotropy. There has to be at least one singular orbit, since the fixed point set $M^{U(1)}$ cannot be empty. We then discuss the 3 cases, in which the number of singular orbits is 1, 2, or 3, respectively.

Case 1: there is only one singular orbit. Then by the slice theorem a tubular neighborhood of the singular orbit is $V = SU(2) \times_{U(1)} D^4 = (SU(2) \times D^4)/U(1)$, where D^4 is a 4-disk and $U(1)$ acts diagonally on $SU(2)$ factor via right translation and on D^4 via the standard linear action on \mathbb{C}^2 . V is a linear D^4 -bundle over $SU(2)/U(1) = S^2$, with boundary $\partial V = S^3 \times S^3/S^1 = SO(4)/SO(2) = T^1S^3 = S^3 \times S^2$. Thus V is a trivial D^4 -bundle over S^2 . Moreover, we claim that the slice action by $SU(2)$ on $\partial V = S^3 \times S^2$ is group multiplication on the S^3 -factor and trivial on the S^2 -factor. To see this, note that the identification $S^3 \times S^2 \cong T^1S^3$ is given by $(p, v_e) \mapsto (p, pv_e)$, where $v_e \in T_e S^3$ and pv_e is quaternion multiplication. $SU(2)$ acts on T^1S^3 via $a.(p, pv_e) = (ap, apv_e) \mapsto (ap, v_e) \in S^3 \times S^2$, and thus it only acts on the S^3 -factor.

The complement U of V is an $SU(2)$ -bundle over D^3 , which has to be the trivial bundle $SU(2) \times D^3 = S^3 \times D^3$ with $SU(2)$ acting only on the first factor. Thus M is the gluing of $U = S^3 \times D^3$ and $V = S^2 \times D^4$ along their common boundary $S^2 \times S^3$ via the equivariant gluing map $f : S^3 \times S^2 \rightarrow S^3 \times S^2$. f has to take on

the form

$$f(p, q) = (p \cdot g(q), \phi(q)), \quad (p, q) \in S^3 \times S^2, \quad g : S^2 \rightarrow S^3, \quad \phi \in \text{Diffeo}(S^2).$$

Since $\pi_2(S^3) = 0$, g is null-homotopic. There are only 2 homotopy classes of f , depending on whether ϕ is orientation-preserving or reversing. Note that there exists an equivariant orientation-reversing diffeomorphism of $U = SU(2) \times D^3$ given by $(g, p) \mapsto (g, -p)$. If f is orientation reversing, we change the orientation on U equivariantly so that f becomes orientation-preserving. Thus up to change of orientation f is homotopic to the identity map, and $M = U \cup_f V$ is equivariantly diffeomorphic to S^6 .

Case 2: there are two singular orbits. Again a tubular neighborhood V of each singular orbit is $V = SU(2) \times_{U(1)} D^4 = S^2 \times D^4$, and M is the gluing of the 2 copies of V along their common boundary $S^2 \times S^3$ via f . Up to a change of orientation of V , f is homotopic to the identity. Thus the resulting manifold is $S^2 \times S^4$.

Case 3: there are three singular orbits. A neighborhood V' of the singular part is the union of three copies of $S^2 \times D^4$ as in the previous cases. The principal part U' of the manifold is a $SU(2)$ -bundle over S^3 minus 3 points, which is homotopy equivalent to a $SU(2)$ -bundle over $S^2 \vee S^2$. $SU(2)$ principal bundles over S^2 are classified by $\pi_1(SU(2)) = 0$, and thus have to be trivial. So U' is homotopy equivalent to a trivial $SU(2)$ -bundle over $S^2 \vee S^2$. $U' \cap V'$ is diffeomorphic to three copies of $S^3 \times S^2$. M is the gluing of U' and V' along $U' \cap V'$ via three copies of f . Each copy of f could be orientation preserving or reversing. We fix the orientation on U' ,

and change the orientation of a component of V' if the corresponding gluing map is orientation-reversing. In conclusion, up to change of orientation there is only one homotopy class of the gluing map and thus only one diffeomorphism class of M . From Example 3(a) described in Chapter 6, we know the flag manifold $SU(3)/T^2$ admits such an action, thus $M = SU(3)/T^2$. \square

Remark 4.2.4. The $SU(2)$ -actions on S^6 , $S^2 \times S^4$ and $SU(3)/T^2$ in Cases 1,2,3 are all realizable. On S^6 it is the triple suspension of the Hopf action on S^3 . On $S^2 \times S^4$ it is the diagonal action where $SU(2)$ acts as $SO(3)$ on S^2 and acts on S^4 as the suspension of S^3 . On $SU(3)/T^2$ it acts via left multiplication. We do not know though whether $S^2 \times S^4$ admits a metric with positive sectional curvature invariant under the $SU(2)$ -action. The $SU(2)$ actions on the 6-sphere in Case 1 and on $SU(3)/T^2$ in Case 3 preserve positive curvature.

We point out that at this point we have proved Theorem 1.0.1(a). Indeed, when the orbit space M^* is a 3-ball, Theorem 1.0.1(a) reduces to Theorem 4.1.1(4). When M^* is a 3-sphere, from Proposition 4.1.4 and Proposition 4.2.1, singular orbits are all isolated whose number is at most 3. In M^{S^1} : each singular orbit contributes to 1 or 2 S^1 -fixed points. Thus M^{S^1} is a finite set of at most 6 points. Hence $\chi(M) = \chi(M^{S^1}) \leq 6$.

Chapter 5

Unsolved Cases

In this chapter, we summarize the unsolved cases and discuss possible strategies.

In the pictures, the dashed oval indicates $M^* = B^3$ and a single circle indicates $M^* = S^3$. Note that the center c has to be the soul point by Theorem 2.0.9.

1. **Figure 5.1a.** ∂M^* has isotropy $SO(2)$ and $\text{int}(M^*)$ has trivial isotropy. From Corollary V.6.2, page 257 of [Bre72], we know such G-spaces are parametrized by

$$[B, (N(H) \cap N(K)) \setminus N(H)]/\pi_0\left(\frac{N(H)}{H}\right) = \pi_2(\mathbb{RP}^2) = \mathbb{Z}.$$

Two examples of such G-spaces are \mathbb{CP}^3 and $S^2 \times S^4$. The $SO(3)$ -action on \mathbb{CP}^3 is the one described in Example 2(b) in Chapter 6, while the action on $S^2 \times S^4$ is diagonal with standard $SO(3)$ -action on S^2 -factor and double suspension on S^4 -factor. We suspect the G-spaces are oriented S^2 -bundles over S^4 , which are classified by the first Pontryagin class. But we do not

know how to construct the desired action on other S^2 -bundles over S^4 besides $S^2 \times S^4$ and \mathbb{CP}^3 . We also do not know whether $S^2 \times S^4$ with this $SO(3)$ -action admits invariant metric with positive sectional curvature.

- 2. Figure 5.1b.** $\partial M^* \setminus \{N, S\}$ and c have isotropy $SO(2)$; N is a fixed point; S has isotropy $O(2)$; the open interval connecting S and c has isotropy $\mathbb{Z}/2$; the rest has trivial isotropy. Example 2(c) of Chapter 6 is an example of this type.

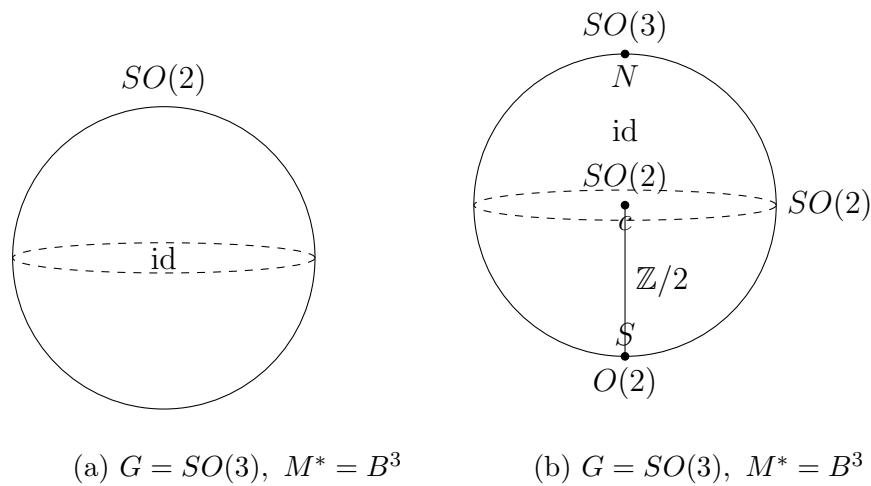


Figure 5.1

- 3. Figure 5.2.** In this case the number of singular orbits is 2 or 3 by Proposition 4.1.4. m, n are positive integers. Examples 2(d) and 3(b) are examples of this type.

- 4. Figure 5.3.** All possible stratifications are depicted in the following pictures: m, n, l are pairwise coprime odd integers. Example 4 of Chapter 6 is an

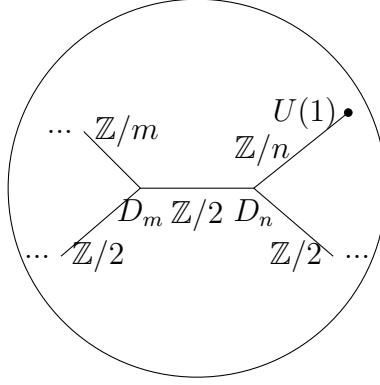


Figure 5.2: $G = SO(3)$, $M^* = S^3$

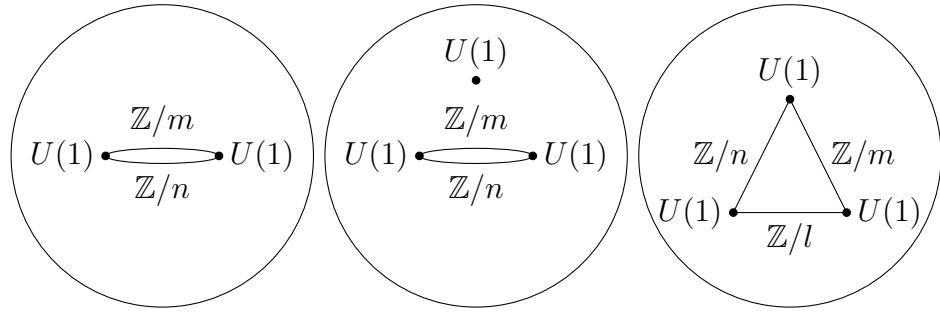


Figure 5.3: $G = SU(2)$, $M^* = S^3$

example of this type.

5. Figure 5.4a. ∂M^* and the center c have isotropy $SO(2)$, and $\text{int}(M^*) \setminus c$ has trivial isotropy. We have no example of such actions.

6. Figure 5.4b. $\partial M^* \setminus \{N, S\}$ and c have isotropy $SO(2)$, N, S are fixed points, and $\text{int}(M^*) \setminus c$ has trivial isotropy. We have no example of such actions.

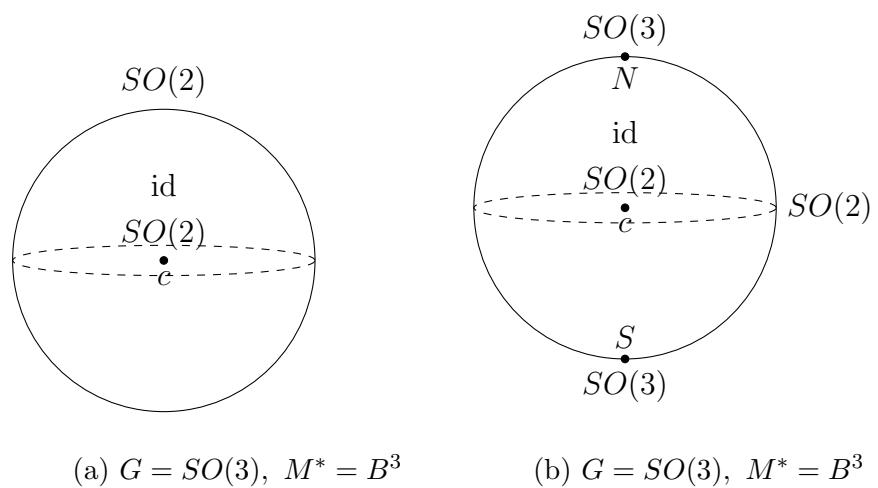


Figure 5.4

Chapter 6

Explicit Examples of G-actions

In this chapter we list all known examples of isometric $SU(2)$, $SO(3)$ -actions on the known examples of positively curved 6-manifolds, namely

$$S^6, \mathbb{CP}^3, SU(3)/T^2, SU(3)//T^2,$$

and depict the stratification of M^* . For S^6 and \mathbb{CP}^3 we list all linear actions. For the known positively curved metrics on $SU(3)/T^2$ and $SU(3)//T^2$, the full isometry group was determined in [GSZ06] and one easily sees that the only isometric actions are the ones described below.

1. Actions on S^6 . Note that all known actions on S^6 are classified.

- (a) **Figure 6.1a.** This action is given by $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y})$, $A \in SU(2)$, $\vec{x} \in \mathbb{R}^4$, $\vec{y} \in \mathbb{R}^3$, $(\vec{x}, \vec{y}) \in S^6$. The action on the \vec{x} -component comes from the real 4-dim irrep of $SU(2)$, i.e. the realification of the standard $SU(2)$ -action on \mathbb{C}^2 , and the action on \vec{y} -component comes from the standard

$SO(3)$ -action on \mathbb{R}^3 . Actions of this type are classified. See Theorem

1.0.2.

- (b) **Figure 6.1b.** G acts on the first 4 coordinates and fixes the last 3 coordinates. ∂M^* consists of fixed points, and the interior has trivial isotropy. Actions of this type are fixed point homogeneous and thus are classified.
- (c) $G = SO(3)$, $M^* = B^4$. G acts on the first 3 coordinates of $S^6 \subset \mathbb{R}^7$ via rotation and fixes the last 4 coordinates. ∂M^* consists of fixed points, and the interior consists of principal orbits with $SO(2)$ -isotropy. This is the only case with $\dim(M^*) = 4$. Actions of this type are fixed point homogeneous and thus are classified.
- (d) **Figure 6.1c.** G acts on the first 5 coordinates via the unique 5-dimensional real representation of $SO(3)$ and fixes the last 2 coordinates. The equator of ∂M^* consists of fixed points, and the two boundary faces corresponding to the two open hemi-spheres have $O(2)$ -isotropy. The interior of M^* consists of principal orbits with isotropy $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Actions of this type have fixed point cohomogeneity one and thus are classified.
- (e) **Figure 6.1d.** This action is given by $A(\vec{x}, \vec{y}, z) = (A\vec{x}, A\vec{y}, z)$, $A \in SO(3)$, $\vec{x}, \vec{y} \in \mathbb{R}^3$, $z \in \mathbb{R}$, $(\vec{x}, \vec{y}, z) \in S^6$. Actions of this type are classified. See Theorem 1.0.3.

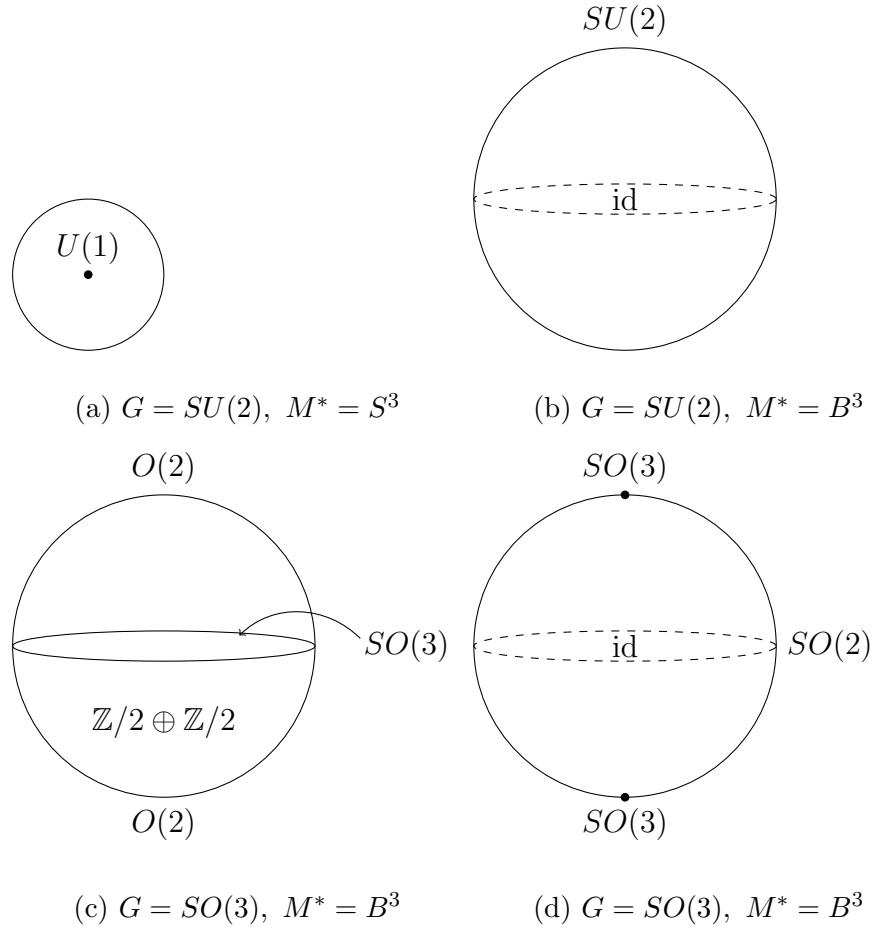


Figure 6.1: $M = S^6$

2. Actions on \mathbb{CP}^3 :

(a) **Figure 6.2a.** A linear $SU(2)$ -action on \mathbb{CP}^3 , acting on the first 2 homogeneous coordinates and fixing the last 2 homogeneous coordinates. $M^* = B^3$. $\partial M^* = S^2$ consists of fixed points, and the interior minus the center has trivial isotropy. The center has $U(1)$ -isotropy, represented by $[x, y, 0, 0] \in \mathbb{CP}^3$. Actions of this type are fixed point homogeneous.

(b) **Figure 6.2b.** This action is induced from one $SU(2)$ -action. Let $A \in$

$SU(2)$ act on \mathbb{CP}^3 via $A(\vec{x}, \vec{y}) = (A\vec{x}, A\vec{y}), \vec{x}, \vec{y} \in \mathbb{C}^2$. This action is ineffective since $-Id \in SU(2)$ acts trivially, thus descends to an $SO(3)$ -action. The interior of M^* consists of principal orbits, and ∂M^* consists of singular $SO(2)$ -orbits.

- (c) **Figure 6.2c.** This action is given by $A(z_1 : z_2 : z_3 : z_4) = (A(z_1 : z_2 : z_3)^T : z_4), A \in SO(3), (z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^3$.
- (d) **Figure 6.2d.** The irreducible representation of $SU(2)$ on \mathbb{C}^4 induces an action on \mathbb{CP}^3 , which is ineffective with kernel $\mathbb{Z}/2$ and descends to $SO(3)$.

3. Actions on $SU(3)/T^2$:

- (a) **Figure 6.3a.** This action is given by left multiplication. Actions of this type are classified. See Theorem 1.0.2.
- (b) **Figure 6.3b.** This action is given by left multiplication.

- 4. **An action on $SU(3)//T^2$.** Recall that the description of the biquotient is given by $SU(3)//T^2 = (z, w, zw) \setminus SU(3)/(1, 1, z^2w^2)^{-1}, z, w \in S^1$. $G = SU(2), M^* = S^3$. $SU(2)$ acts from the right as the first 2 block of $SU(3)$, commuting with the T^2 -action. The orbit strata are indicated in **Figure 6.4**. A computation, using Mayer-Vietoris sequence, shows that G-spaces of this type have the same cohomology groups as $SU(3)//T^2$, that is, $H^0 = H^6 = \mathbb{Z}, H^2 = H^4 = \mathbb{Z} \oplus \mathbb{Z}, H^{2i+1} = 0$.

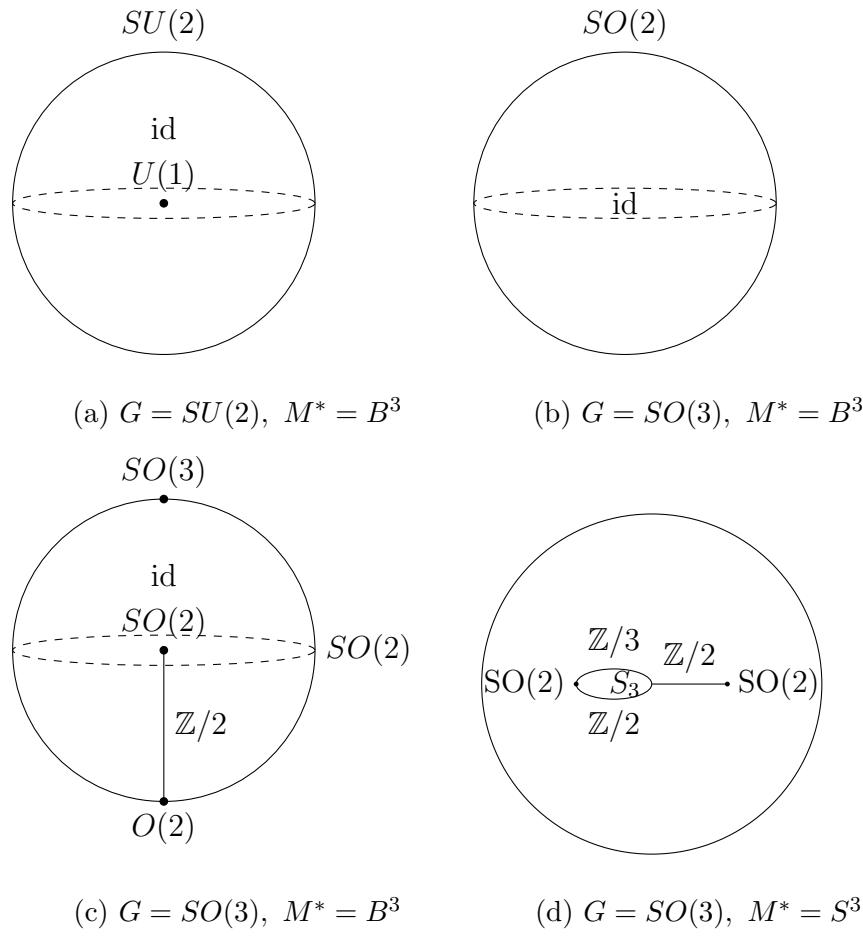
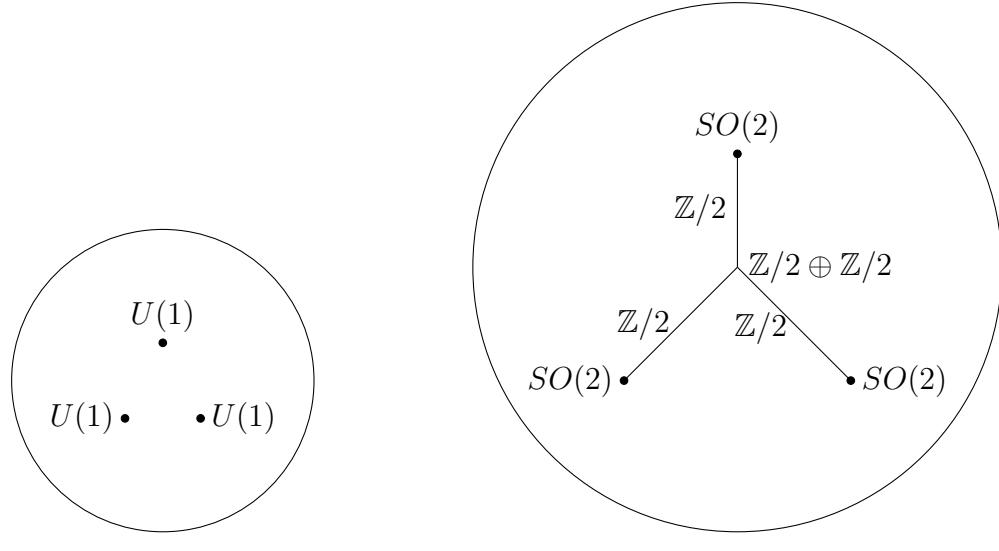


Figure 6.2: $M = \mathbb{CP}^3$



(a) $G = SU(2), M^* = S^3$

(b) $G = SO(3), M^* = S^3$

Figure 6.3: $M = SU(3)/T^2$

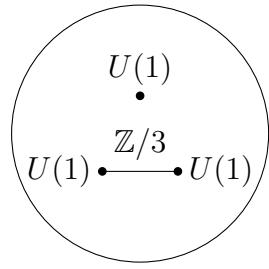


Figure 6.4: $G = SU(2), M^* = S^3$

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