Problems and Solutions

Preliminary Examination, Part I

Monday, August 26, 2019 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. a) Show that there is no real polynomial $p(x)$ so that $\cos x = p(x)$ for all real x.

5 points

Solution: Some properties of $u(x) := \cos x$:

i). $\cos x$ is periodic but not a constant
ii). infinitely many zeros
iii). $|\cos x| \le 1$
iv). $u'' = -u$

The only polynomial that satisfies ii). is $p(x) \equiv 0$.
$p(x)$ is unbounded unless $p(x) \equiv$ constant
The derivative of a polynomial has a lower degree - which violates iv). This also shows that $\cos x$ is not a polynomial on a small interval.

b) Show that $\cos x$ is not a rational function, that is, there are no polynomials $p(x)$ and $q(x)$ so that $\cos x = \frac{p(x)}{q(x)}$ for all real x.

5 points

Solution: If $\cos x = \frac{p(x)}{q(x)}$, then:
$p(x)$ has infinitely many zeros.

If degree $(p) >$ degree (q), then $\cos x$ would be unbounded, while if degree $(p) \le$ degree (q), then $\cos x$ would converge to a constant as $x \to \pm\infty$, contradicting the periodicity.

Write $r(x) = p(x)/q(x)$ and let $\mathrm{DEG}\,(r) :=$ degree $(p) -$ degree (q). If $q(x)$ is not a constant, then $\mathrm{DEG}\,(r') < \mathrm{DEG}\,(r)$ so $r(x)$ could not satisfy property iv). This also shows that $\cos x$ is not a rational function on a small interval.

2. Classify finite groups of order 45 (up to isomorphism).

10 points

Solution: Let G be a group of order 45. From Sylow's theorem, the Sylow 3-group and 5 group are unique. Denote them by H and K, which are both normal subgroups of G and $H \cap K = \{1\}$. So $HK \cong H \times K$ and $G = HK$.

Since $|H| = 9$, so $H \cong \mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Thus $G \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $\lim_{t\to\infty} f(t) = 0$. Show that
$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)\,dt = 0.$$

10 points

Solution: Pick T_0 so that if $t > T_0$ then $|f(t)| < \epsilon$. Say $T > T_0$. Then

$$\begin{aligned}\left|\frac{1}{T}\int_0^T f(t)\,dt\right| \leq& \frac{1}{T}\int_0^T |f(t)|\,dt \\ =& \frac{1}{T}\int_0^{T_0} |f(t)|\,dt + \frac{1}{T}\int_{T_0}^T |f(t)|\,dt \\ <& \frac{1}{T}\int_0^{T_0} |f(t)|\,dt + \frac{T-T_0}{T}\epsilon = A + B\end{aligned}$$

Clearly $B < \epsilon$. To show that $A < \epsilon$ for T large, let $M := \max_{0\leq t\leq T_0}|f(t)|$. Then for sufficiently large T
$$A \leq \frac{MT_0}{T} < \epsilon.$$

Alternate: Strange – but short. Let $g(t) := f(t) + 1$ so $\lim_{t\to\infty} g(t) = 1$. Then

$$\frac{1}{T}\int_0^T g(t)\,dt = 1 + \frac{1}{T}\int_0^T f(t)\,dt$$

But by l'Hôpital, $\displaystyle\frac{1}{T}\int_0^T g(t)\,dt \to 1$. Thus $\displaystyle\frac{1}{T}\int_0^T f(t)\,dt \to 0$.

4. Let $\mathcal{P}_n$ be the linear space of polynomials $p(x) \in \mathbb{R}[x]$ of degree at most n and let $L : \mathcal{P}_n \to \mathcal{P}_n$ be the linear map defined by $Lu := u'' + bu' + cu$, where b and c are constants. Assume $c \neq 0$.

   a) Find all $p \in \mathcal{P}_n$ that satisfy $Lp = 0$.

5 points

   Solution: Claim: $p = 0$. Say $p(x) = ax^k +$ lower order and where $a \neq 0$. Then $Lp = acx^k +$ lower order. Since $c \neq 0$, $Lp = 0$ implies that $a = 0$, a contradiction.

   b) Show that for every polynomial $q(x) \in \mathcal{P}_n$ there is one (and only one) solution $p(x) \in \mathcal{P}_n$ of $Lp = q$. In other words, for $c \neq 0$, the map $L : \mathcal{P}_n \to \mathcal{P}_n$ is invertible. [Note: You are not being asked to find a formula for p.]

5 points

   Solution: Method 1. By part a), $L\mathcal{P}_n \to \mathcal{P}_n$, has $\ker L = 0$. Therefore L is invertible.

   Method 2. Use induction on n. If $n = 0$, since $c \neq 0$, the constant $p = a/c$ satisfies $Lp = a$. Say for any $q \in \mathcal{P}_k$ there is a solution $p \in \mathcal{P}_k$ of $Lp = q$. Let $\hat{q} = ax^{k+1} +$ lower order and seek a solution $\hat{p} = (a/c)x^{k+1} +$ lower order. Then $L\hat{p} = ax^{k+1} +$ lower order so by the induction hypothesis there is a solution.

Method 3. Rewrite the equation $p'' + bp' + cp = q$ as

$$p + (1/c)[p'' + bp'] = (1/c)q$$

and let $Mu := (1/c)[u'' + bu']$. Then our equation is $(I + M)p = (1/c)q$. But acting on polynomials, $M : \mathcal{P}_k \to \mathcal{P}_{k-1}$ is nilpotent. Thus $I + M$ is invertible so $p = (I + M)^{-1}q$.

5. a) Let f be a continuous function on the interval $\{x \,|\, 1 \le x \le 3\}$. Compute

$$\lim_{n\to\infty} \int_1^3 f(x)e^{-nx}\,dx.$$

[Justify your assertions.]

5 points

SOLUTION: Since $f \in C([1,3])$, it is bounded, so say $|f(x)| \le M$ in $[1,3]$. Then

$$\left|\int_1^3 f(x)e^{-nx}\,dx\right| \le M \int_1^3 e^{-nx}\,dx \le 2Me^{-n} \to 0.$$

ALTERNATE: Observe that the sequence $\lim_{n\to\infty} f(x)e^{-nx} = 0$ *uniformly* on the bounded interval $[1,3]$ so we can interchange limit and integral.

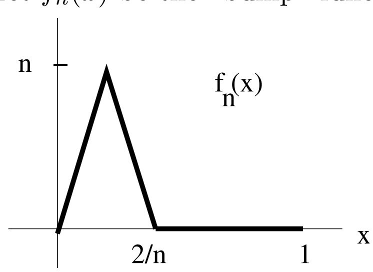
b) Give an example of a sequence of continuous real-valued functions $f_n(x) \ge 0$ with the property $f_n(x) \to 0$ for all $x \in [0,\, 1]$ but

$$\int_0^1 f_n(x)\,dx \ge 1 \qquad \text{for } all\ n = 1, 2, \ldots$$

If you prefer, a clear sketch of a graph will be adequate.

5 points

SOLUTION Let $f_n(x)$ be the "bump" function in the figure.



MORE GENERAL: Let $g \in C([0,1])$ have the properties 1). $g(0) = 0$, 2). $g(x) \ge 0$ for $0 < x < 1$, 3). $g(x) = 0$ for $x \ge 1$, and 4). $\int_0^1 g(x)\,dx = 1$. Then $f_n(x) := ng(nx)$ is an example.

6. a) Let M be a complete metric space. Suppose $K \subset M$ is a compact subset and P is a point in M with $P \notin K$. Show there is a point $Q \in K$ that is closest to P, that is,

$$d(P,Q) = \inf_{x\in K} d(P,x).$$

5 points

SOLUTION: Let $h(x) := d(P, x)$. Because $d(P, x) \leq d(P, y) + d(x, y)$, then $|h(x) - h(y)| \leq d(x, y)$ so $h(x)$ is a continuous function of $x \in M$, Since K is compact, there is a point $q \in K$ where h has its minimum value on K.

b) Consider the metric space ℓ_2 of real sequences $\{x = (x_1, x_2, \ldots) \mid x_j \in \mathbb{R}\}$ with norm $|x|^2 = \sum_j x_j^2 < \infty$, inner product $\langle x, y\rangle = x_1 y_1 + x_2 y_2 + \cdots$, and with metric given by $d(x, y) := |x - y|$.

Let $Q \subset \ell_2$ be the (standard) set of unit orthonormal vectors $\{e_j,\ j = 1, 2, 3, \ldots\}$, where $e_1 = (1, 0, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$, $\ldots$, $e_k = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the k^{th} slot.

Is the set Q closed in ℓ_2?, Is it bounded? Is it compact? Justify your assertions.

closed 1 pt

SOLUTION: Q is closed: (i). The complement of Q ia open, (ii). Q has no limit points since $|e_i - e_j| = \sqrt{2}$ for $i \neq j$.

bndd 1 pt

Q is bounded since $|e_j| = 1$ for $j = 1, 2, \ldots$.

compct 3 pt

However, Q is *not* compact. Several proofs:

PROOF 1. The balls $B_j = \{x \in \ell_2 : |x - e_j| < 1\}, j = 1, 2, \ldots$ are an open cover of Q. Since each of the B_j'a contains only one point of Q, there is no finite sub-cover.

PROOF 2. Since $|e_i - e_j| = \sqrt{2}$ for $i \neq j$, the sequence $e_j,\ j = 1, 2, \ldots$ has no convergent subsequence.

PROOF 3. Since the continuous function $f : Q \to R$ defined by $f(e_k) = k$ for $k = 1, 2, \ldots$ is unbounded, Q could not be compact.

Similarly, we construct a continuous function $g : Q \to \mathbb{R}$ that does not take on its upper bound. Let $g(e_k) = 1 - \frac{1}{k}$, $k = 1, 2, \ldots$. Since $|e_i - e_j| = \sqrt{2}$ for all $i \neq j$, g is continuous on Q. Clearly $\sup_{x \in Q} g(x) = 1$ but there is no $p \in Q$ where $g(p) = 1$.

REMARK: Generalizing Proof 2, F. Riesz showed that the closed unit ball in any normed linear space is compact if and only if the space is finite dimensional.

Preliminary Examination, Part II

Monday, August 26, 2019 1:30-4:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight of 10 points.

7. Let $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with smooth boundary $\partial\Omega$. Suppose $\mathbf{F}(x)$ is an infinitely differentiable vector field defined for $x \in \mathbb{R}^3$, and $u(x)$ is an infinitely differentiable real-valued function defined for $x \in \mathbb{R}^3$.

   NOTATION: ∇u is the gradient of u and $\nabla\cdot\mathbf{F}$ is the divergence of $\mathbf{F}$.

   a) Verify the formula for the derivative of the product

   $$\nabla\cdot\big(u(x)\mathbf{F}(x)\big) = \nabla u\cdot\mathbf{F} + u\,\nabla\cdot\mathbf{F}. \tag{1}$$

2 points

   SOLUTION: To verify this write $x = (x_1, x_2, x_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$. Then $u\mathbf{F} = (uF_1, uF_2, uF_3)$ so

   $$\begin{aligned}\nabla\cdot\big(u(x)\mathbf{F}(x)\big) =& \frac{\partial(uF_1)}{\partial x_1} + \frac{\partial(uF_2)}{\partial x_2} + \frac{\partial(uF_3)}{\partial x_3} \\ =& \frac{\partial u}{\partial x_1}F_1 + u\frac{\partial F_1}{\partial x_1} + \frac{\partial u}{\partial x_2}F_2 + u\frac{\partial F_2}{\partial x_2} + \frac{\partial u}{\partial x_3}F_3 + u\frac{\partial F_3}{\partial x_3} \\ =& \nabla u\cdot\mathbf{F} + u\,\nabla\cdot\mathbf{F}\end{aligned}$$

   b) Use Part a) to obtain the generalization of *integration by parts*:

   $$\iiint_\Omega u\,\nabla\cdot\mathbf{F}\,dV = \iint_{\partial\Omega} u\,\mathbf{F}\cdot\mathbf{n}\,dA - \iiint_\Omega \nabla u\cdot\mathbf{F}\,dV, \tag{2}$$

   where dV is the element of volume on Ω, dA the element of area on $\partial\Omega$, and $\mathbf{n}$ a unit outer normal vector field on $\partial\Omega$. [Hint: Use the divergence theorem].

4 points

   SOLUTION: The divergence theorem applied to the vector field $u\mathbf{F}$ gives:

   $$\iiint_\Omega \nabla\cdot\big(u\mathbf{F}\big)\,dV = \iint_{\partial\Omega} u\,\mathbf{F}\cdot\mathbf{n}\,dA.$$

Now use (1) in the integral on the left to obtain

$$\iiint_\Omega u\,\nabla\cdot\mathbf{F}\,dV + \iiint_\Omega \nabla u\cdot\mathbf{F}\,dV = \iint_{\partial\Omega} u\,\mathbf{F}\cdot\mathbf{n}\,dA \tag{3}$$

which is just (2).

c) In the special case of $\mathbf{F} = \nabla u$, the equation (2) is the identity

$$\iiint_\Omega u\,\nabla\cdot\nabla u\,dV = \iint_{\partial\Omega} u\,\nabla u\cdot\mathbf{n}\,dA - \iiint_\Omega |\nabla u|^2\,dV. \tag{4}$$

Use this to show that if $\nabla\cdot\nabla u = 0$ in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in all of Ω. [Remark: $\nabla\cdot\nabla u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$, the Laplacian, is often written as Δu.]

4 points

SOLUTION: If $\nabla\cdot\nabla u = 0$ in Ω and $u = 0$ on $\partial\Omega$, then (4) inplies that $\iiint_\Omega |\nabla u|^2\,dV = 0$. Hence $\nabla u = 0$ in Ω. Consequently $u =$ constant. But since $u = 0$ on $\partial\Omega$, then $u(x) \equiv 0$ in Ω.

8. Let $\vec{x},\, \vec{y} \in \mathbb{R}^n$ with the usual inner product which we write as $\langle x,\, y\rangle$ (the notation $\vec{x}\cdot\vec{y}$ is also often used). Also, we write the norm as $|\vec{x}| = \sqrt{\langle \vec{x},\, \vec{x}\rangle}$.

Let A be a real symmetric $n\times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Show that

$$\langle \vec{x},\, A\vec{x}\rangle \ge \lambda_1 |\vec{x}|^2 \qquad \text{for all} \quad \vec{x}.$$

10 points

SOLUTION: Version 1. Since A is a real symmetric matrix, then $\mathbb{R}^n$ has an orthonormal basis of eigenvectors $\vec{v}_1,\, \dots,\, \vec{v}_n$ of A, so $A\vec{v}_j = \lambda_j \vec{v}_j$, $j = 1, \dots n$.

Write $\vec{x} \in \mathbb{R}^n$ in this basis:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Using the orthonormality of the $\vec{v}_j$:

$$|\vec{x}|^2 = c_1^2 + c_2^2 + \cdots + c_n^2.$$

Also,

$$A\vec{x} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_n\lambda_n\vec{v}_n$$

so, again using the orthonormality of the $\vec{v}_j$,

$$\begin{aligned}\langle \vec{x},\, A\vec{x}\rangle &= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 \\ &\ge \lambda_1 (c_1^2 + c_2^2 + \cdots + c_n^2) \\ &= \lambda_1 |\vec{x}|^2.\end{aligned}$$

Version 2. This direct proof does not use the characteristic polynomial. Historically, since it concerns the geometric principal axes of the conic $\langle \vec{x},\, A\vec{x}\rangle = 1$, it probably predates Version 1, particularly for positive definite matrices. [It also generalizes to give the eigenvalues of the Laplacian.]

The quadratic $\varphi(\vec{x}) := \langle \vec{x}, A\vec{x} \rangle$ is a continuous function on the unit sphere $S = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| = 1\}$. Since S is compact, there is a point $\vec{v} \in S$ where $\varphi(x)$ has its minimum:

$$\mu := \langle \vec{v}, A\vec{v} \rangle = \min_{\vec{x} \neq 0} \frac{\langle \vec{x}, A\vec{x} \rangle}{|\vec{x}|^2}.$$

We first show that $A\vec{v} = \mu\vec{v}$ so μ is an eigenvalue of A. Let $\vec{z} \in \mathbb{R}^n$ and $\vec{x} = \vec{v} + t\vec{z}$ for small $t \in \mathbb{R}$ (so $\vec{x} \neq 0$). Let

$$h(t) := \varphi(\vec{x}) = \frac{\langle \vec{v} + t\vec{z}, A(\vec{v} + t\vec{z}) \rangle}{|\vec{v} + t\vec{z}|^2}.$$

Then by definition of $\vec{v}$, $h(t)$ has its minimum at $t = 0$ so $h(t) \geq h(0) = \mu$. Thus $h'(0) = 0$ for all possible choices of $\vec{z}$. We compute $h'(0)$ by routine calculus:

$$\begin{aligned} h'(0) =& \frac{\langle \vec{z}, A\vec{v} \rangle + \langle \vec{v}, A\vec{z} \rangle}{|\vec{v}|^2} - \frac{\langle \vec{v}, A\vec{v} \rangle 2 \langle \vec{v}, \vec{z} \rangle}{|\vec{v}|^4} \\ =& 2\langle A\vec{v}, \vec{z} \rangle - 2\mu \langle \vec{v}, \vec{z} \rangle \\ =& 2\langle A\vec{v} - \mu\vec{v}, \vec{z} \rangle \end{aligned}.$$

Since $h'(0) = 0$, then $\langle A\vec{v} - \mu\vec{v}, \vec{z} \rangle = 0$ for all vectors $\vec{z} \in \mathbb{R}^n$. Consequently $A\vec{v} - \mu\vec{v} = 0$ so $\vec{v}$ is an eigenvector with eigenvalue μ. Note that any eigenvector $\vec{u}$ of A with eigenvalue λ satisfies $\langle \vec{u}, A\vec{u} \rangle = \lambda|\vec{u}|^2 \geq \mu|\vec{u}|^2$ so μ is the smallest eigenvalue.

[In Version 2 to show that $\vec{v}$ is an eigenvector of A we could also have used Lagrange multipliers.]

9. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with the properties $f(0) = 3$, $f(1) = 1$, and $f(3) = 5$. Find an explicit positive real number A such that there exists a real number c with $0 < c < 3$ such that $f''(c) \geq A$.

10 points

SOLUTION: By the mean value theorem there are points $0 < c_1 < 1$ and $1 < c_2 < 3$ where

$$f'(c_1) = \frac{1-3}{1-0} = -2 \qquad \text{and} \qquad f'(c_2) = \frac{5-1}{3-1} = 2.$$

Applying the mean value theorem to the interval $[c_1, c_2]$ there is a point $c_1 < c < c_2$ where

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{2-(-2)}{c_2 - c_1} > \frac{4}{3}.$$

We can therefore take $A = 4/3$ (or any smaller positive value).

[REMARK: For the optimal value of A let $p(x) = \alpha x^2 + \beta x + \gamma$ be the unique quadratic polynomial passing through these three points. Then $A_{\text{optimal}} = 2\alpha$.]

10. Let R denote the ring $\dfrac{\mathbb{Z}[x]}{(2x^2+2x+1)}$. Prove that R is an integral domain.

10 points SOLUTION: To show that R is a domain, we need to show that the ideal

$$(2x^2+2x+1)$$

is prime. Since $\mathbb{Z}$ is a UFD, Gauss's lemma tells us that $\mathbb{Z}[x]$ is a UFD. As a result, it suffices to show that $2x^2+2x+1$ is irreducible in $\mathbb{Z}[x]$. Since $\mathbb{Z}[x] \subset \mathbb{Q}[x]$, it suffices to show $2x^2+2x+1$ is irreducible in $\mathbb{Q}[x]$. Since $2x^2+2x+1$ is of degree 2, this is tantamount to asking whether $2x^2+2x+1$ has a linear factor in $\mathbb{Q}[x]$, that is whether $2x^2+2x+1$ has any root in $\mathbb{Q}$. Since the discriminant of the polynomial equals -4, the quadratic polynomial $2x^2+2x+1$ has no roots in $\mathbb{Q}$. This completes the proof.

11. Find an integer N so that $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{N} > 100$.

10 points SOLUTION: Use the geometric idea of the integral test:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{N} > \int_1^{N+1} \frac{1}{x}\,dx = \ln(N+1).$$

Thus pick $\ln(N+1) > 100$, that is, $N+1 > e^{100}$; we may take N to be the greatest integer in e^{100}.

ALTERNATE A direct grouping of terms. Let $N = 2^k$. Then

$$\begin{aligned}
S_N :=&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{N}\\
=&1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{k-1}+1}+\cdots+\frac{1}{2^k}\right)\\
>&1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^k}+\cdots+\frac{1}{2^k}\right)\\
=&1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}\\
=&1+k\frac{1}{2}.
\end{aligned}$$

Pick k so that $1+(k/2) = 100$, that is, $k = 198$. Then $S_{2^{198}} > 100$.

12. Let A be an $n \times n$ real or complex matrix.

a) Show that $\ker A^j \subset \ker A^{j+1}$. If $\ker A^k = \ker A^{k+1}$ for some k, show that $\ker A^j = \ker A^k$ for all $j \ge k$.

5 points SOLUTION: If $A^j x = 0$ then $A^{j+1}x = A(A^j x) = 0$.

For the second part, by induction, say $x \in \ker A^{k+2}$. Then $0 = A^{k+2}x = A^{k+1}(Ax)$ so $Ax \in \ker A^{k+1}$. But then $Ax \in \ker A^k$, that is, $A^{k+1}x = 0$.

b) Say A is a nilpotent 5×5 matrix. Is it true that $A^5 = 0$? Proof or counterexample.

5 points

SOLUTION (not using part a). Since A is nilpotent, it satisfies $A^k = 0$ for some k. The characteristic polynomial, λ^5, is divisible by the minimal polynomial, $p(\lambda) = \lambda^m$, of A. Thus $m \leq 5$, and A satisfies $A^5 = 0$.

ALTERNATE (this approach uses part a). Note that $\dim \ker A^j$ is strictly increasing until it remains constant. Since $\dim \ker A \leq 5$ and $\dim \ker A \geq 1$, it can only increase 4 times so $A^5 = 0$.