Problems and Solutions
PRELIMINARY EXAMINATION, PART I

Monday, August 26, 2019 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. a) Show that there is no real polynomial \( p(x) \) so that \( \cos x = p(x) \) for all real \( x \).

5 points

**SOLUTION:** Some properties of \( u(x) := \cos x \):

i). \( \cos x \) is periodic but not a constant

ii). infinitely many zeros

iii). \( |\cos x| \leq 1 \)

iv). \( u'' = -u \)

The only polynomial that satisfies ii). is \( p(x) \equiv 0 \).

\( p(x) \) is unbounded unless \( p(x) \equiv \) constant

The derivative of a polynomial has a lower degree - which violates iv). This also shows that \( \cos x \) is not a polynomial on a small interval.

b) Show that \( \cos x \) is not a rational function, that is, there are no polynomials \( p(x) \) and \( q(x) \) so that \( \cos x = \frac{p(x)}{q(x)} \) for all real \( x \).

5 points

**SOLUTION:** If \( \cos x = \frac{p(x)}{q(x)} \), then:

\( p(x) \) has infinitely many zeros.

If \( \deg(p) > \deg(q) \), then \( \cos x \) would be unbounded, while if \( \deg(p) \leq \deg(q) \), then \( \cos x \) would converge to a constant as \( x \to \pm \infty \), contradicting the periodicity.

Write \( r(x) = p(x)/q(x) \) and let \( \deg(r) := \deg(p) - \deg(q) \). If \( q(x) \) is not a constant, then \( \deg(r') < \deg(r) \) so \( r(x) \) could not satisfy property iv). This also shows that \( \cos x \) is not a rational function on a small interval.

2. Classify finite groups of order 45 (up to isomorphism).

10 points

**SOLUTION:** Let \( G \) be a group of order 45. From Sylow's theorem, the Sylow 3-group and 5 group are unique. Denote them by \( H \) and \( K \), which are both normal subgroups of \( G \) and \( H \cap K = \{1\} \). So \( HK \cong H \times K \) and \( G = HK \).

Since \( |H| = 9 \), so \( H \cong \mathbb{Z}/9\mathbb{Z} \) or \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \).

Thus \( G \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \) or \( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \).
3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with the property that \( \lim_{t \to \infty} f(t) = 0 \). Show that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = 0.
\]

10 points

**SOLUTION:** Pick \( T_0 \) so that if \( t > T_0 \) then \(|f(t)| < \epsilon\). Say \( T > T_0 \). Then

\[
\left| \frac{1}{T} \int_0^T f(t) \, dt \right| \leq \frac{1}{T} \int_0^{T_0} |f(t)| \, dt + \frac{1}{T} \int_{T_0}^T |f(t)| \, dt < \frac{1}{T} \int_0^{T_0} |f(t)| \, dt + \frac{T - T_0}{T} \epsilon = A + B
\]

Clearly \( B < \epsilon \). To show that \( A < \epsilon \) for \( T \) large, let \( M := \max_{0 \leq t \leq T_0} |f(t)| \). Then for sufficiently large \( T \)

\[
A \leq \frac{MT_0}{T} < \epsilon.
\]

**Alternate:** Strange – but short. Let \( g(t) := f(t) + 1 \) so \( \lim_{t \to \infty} g(t) = 1 \). Then

\[
\frac{1}{T} \int_0^T g(t) \, dt = 1 + \frac{1}{T} \int_0^T f(t) \, dt
\]

But by l’Hôpital, \( \frac{1}{T} \int_0^T g(t) \, dt \to 1 \). Thus \( \frac{1}{T} \int_0^T f(t) \, dt \to 0 \).

4. Let \( \mathcal{P}_n \) be the linear space of polynomials \( p(x) \in \mathbb{R}[x] \) of degree at most \( n \) and let \( L : \mathcal{P}_n \to \mathcal{P}_n \) be the linear map defined by \( Lu := u'' + bu' + cu \), where \( b \) and \( c \) are constants. Assume \( c \neq 0 \).

a) Find all \( p \in \mathcal{P}_n \) that satisfy \( Lp = 0 \).

5 points

**SOLUTION:** Claim: \( p = 0 \). Say \( p(x) = ax^k + \) lower order and where \( a \neq 0 \). Then

\( Lp = acx^k + \) lower order. Since \( c \neq 0 \), \( Lp = 0 \) implies that \( a = 0 \), a contradiction.

b) Show that for every polynomial \( q(x) \in \mathcal{P}_n \) there is one (and only one) solution \( p(x) \in \mathcal{P}_n \) of \( Lp = q \). In other words, for \( c \neq 0 \), the map \( L : \mathcal{P}_n \to \mathcal{P}_n \) is invertible. [NOTE: You are not being asked to find a formula for \( p \).]

5 points

**SOLUTION:** Method 1. By part a), \( L\mathcal{P}_n \to \mathcal{P}_n \), has ker \( L = 0 \). Therefore \( L \) is invertible.

Method 2. Use induction on \( n \). If \( n = 0 \), since \( c \neq 0 \), the constant \( p = a/c \) satisfies \( Lp = a \). Say for any \( q \in \mathcal{P}_k \) there is a solution \( p \in \mathcal{P}_k \) of \( Lp = q \). Let \( \tilde{q} = ax^{k+1} + \) lower order and seek a solution \( \tilde{p} = (a/c)x^{k+1} + \) lower order. Then \( L\tilde{p} = ax^{k+1} + \) lower order so by the induction hypothesis there is a solution.
Method 3. Rewrite the equation $p'' + bp' + cp = q$ as
\[ p + (1/c)[p'' + bp'] = (1/c)q \]
and let $Mu := (1/c)[u'' + bu']$. Then our equation is $(I + M)p = (1/c)q$. But acting on polynomials, $M : \mathcal{P}_k \to \mathcal{P}_{k-1}$ is nilpotent. Thus $I + M$ is invertible so $p = (I + M)^{-1}q$.

5. a) Let $f$ be a continuous function on the interval $\{x \mid 1 \leq x \leq 3\}$. Compute
\[ \lim_{n \to \infty} \int_1^3 f(x)e^{-nx} \, dx. \]
[Justify your assertions.]

5 points
SOLUTION: Since $f \in C([1,3])$, it is bounded, so say $|f(x)| \leq M$ in $[1,3]$. Then
\[ \left| \int_1^3 f(x)e^{-nx} \, dx \right| \leq M \int_1^3 e^{-nx} \, dx \leq 2Me^{-n} \to 0. \]

ALTERNATE: Observe that the sequence $\lim_{n \to \infty} f(x)e^{-nx} = 0$ uniformly on the bounded interval $[1,3]$ so we can interchange limit and integral.

b) Give an example of a sequence of continuous real-valued functions $f_n(x) \geq 0$ with the property $f_n(x) \to 0$ for all $x \in [0,1]$ but
\[ \int_0^1 f_n(x) \, dx \geq 1 \quad \text{for all } n = 1, 2, \ldots \]
If you prefer, a clear sketch of a graph will be adequate.

5 points
SOLUTION Let $f_n(x)$ be the “bump” function in the figure.

More General: Let $g \in C([0,1])$ have the properties 1). $g(0) = 0$, 2). $g(x) \geq 0$ for $0 < x < 1$, 3). $g(x) = 0$ for $x \geq 1$, and 4). $\int_0^1 g(x) \, dx = 1$. Then $f_n(x) := ng(nx)$ is an example.

6. a) Let $M$ be a complete metric space. Suppose $K \subset M$ is a compact subset and $P$ is a point in $M$ with $P \notin K$. Show there is a point $Q \in K$ that is closest to $P$, that is,
\[ d(P, Q) = \inf_{x \in K} d(P, x). \]
Let \( h(x) := d(P, x) \). Because \( d(P, x) \leq d(P, y) + d(x, y) \), then \( |h(x) - h(y)| \leq d(x, y) \) so \( h(x) \) is a continuous function of \( x \in M \). Since \( K \) is compact, there is a point \( q \in K \) where \( h \) has its minimum value on \( K \).

b) Consider the metric space \( \ell_2 \) of real sequences \( \{x = (x_1, x_2, \ldots) | x_j \in \mathbb{R}\} \) with norm \( |x|^2 = \sum_j x_j^2 < \infty \), inner product \( \langle x, y \rangle = x_1y_1 + x_2y_2 + \cdots \), and with metric given by \( d(x, y) := |x - y| \).

Let \( Q \subset \ell_2 \) be the (standard) set of unit orthonormal vectors \( \{e_j, j = 1, 2, 3, \ldots \} \), where \( e_1 = (1, 0, 0, 0, \ldots) \), \( e_2 = (0, 1, 0, 0, \ldots) \), \( e_3 = (0, 0, 1, 0, \ldots) \), \ldots, \( e_k = (0, \ldots, 0, 1, 0, \ldots) \) with 1 in the \( k \)th slot.

Is the set \( Q \) closed in \( \ell_2 \)? Is it bounded? Is it compact? Justify your assertions.

**Solution:** \( Q \) is closed: \( \text{i}. \) The complement of \( Q \) ia open, \( \text{ii}. \) \( Q \) has no limit points since \( |e_i - e_j| = \sqrt{2} \) for \( i \neq j \).

\( Q \) is bounded since \( |e_j| = 1 \) for \( j = 1, 2, \ldots \)

However, \( Q \) is not compact. Several proofs:

**Proof 1.** The balls \( B_j = \{x \in \ell_2 : |x - e_j| < 1\}, j = 1, 2, \ldots \) are an open cover of \( Q \). Since each of the \( B_j \)'s contains only one point of \( Q \), there is no finite sub-cover.

**Proof 2.** Since \( |e_i - e_j| = \sqrt{2} \) for \( i \neq j \), the sequence \( e_j, j = 1, 2, \ldots \) has no convergent subsequence.

**Proof 3.** Since the continuous function \( f : Q \to R \) defined by \( f(e_k) = k \) for \( k = 1, 2, \ldots \) is unbounded, \( Q \) could not be compact.

Similarly, we construct a continuous function \( g : Q \to \mathbb{R} \) that does not take on its upper bound. Let \( g(e_k) = 1 - \frac{1}{k}, k = 1, 2, \ldots \). Since \( |e_i - e_j| = \sqrt{2} \) for all \( i \neq j \), \( g \) is continuous on \( Q \). Clearly \( \sup_{x \in Q} g(x) = 1 \) but there is no \( p \in Q \) where \( g(p) = 1 \).

**Remark:** Generalizing Proof 2, F. Riesz showed that the closed unit ball in any normed linear space is compact if and only if the space is finite dimensional.
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7. Let $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with smooth boundary $\partial \Omega$. Suppose $F(x)$ is an infinitely differentiable vector field defined for $x \in \mathbb{R}^3$, and $u(x)$ is an infinitely differentiable real-valued function defined for $x \in \mathbb{R}^3$.

**NOTATION:** $\nabla u$ is the gradient of $u$ and $\nabla \cdot F$ is the divergence of $F$.

a) Verify the formula for the derivative of the product

$$\nabla \cdot (u(x)F(x)) = \nabla u \cdot F + u \nabla \cdot F.$$  

SOLUTION: To verify this write $x = (x_1, x_2, x_3)$ and $F = (F_1, F_2, F_3)$. Then $uF = (uF_1, uF_2, uF_3)$ so

$$\nabla \cdot (u(x)F(x)) = \frac{\partial (uF_1)}{\partial x_1} + \frac{\partial (uF_2)}{\partial x_2} + \frac{\partial (uF_3)}{\partial x_3}$$

$$= \frac{\partial u}{\partial x_1}F_1 + u\frac{\partial F_1}{\partial x_1} + \frac{\partial u}{\partial x_2}F_2 + u\frac{\partial F_2}{\partial x_2} + \frac{\partial u}{\partial x_3}F_3 + u\frac{\partial F_3}{\partial x_3}$$

$$= \nabla u \cdot F + u \nabla \cdot F$$

b) Use Part a) to obtain the generalization of integration by parts:

$$\iint_{\Omega} u \nabla \cdot F \, dV = \int_{\partial \Omega} uF \cdot n \, dA - \iint_{\Omega} \nabla u \cdot F \, dV,$$  

where $dV$ is the element of volume on $\Omega$, $dA$ the element of area on $\partial \Omega$, and $n$ a unit outer normal vector field on $\partial \Omega$. [Hint: Use the divergence theorem].

SOLUTION: The divergence theorem applied to the vector field $uF$ gives:

$$\iint_{\Omega} \nabla \cdot (uF) \, dV = \int_{\partial \Omega} uF \cdot n \, dA.$$
Now use (1) in the integral on the left to obtain
\[
\iiint_{\Omega} u \nabla \cdot F \, dV + \iiint_{\Omega} \nabla u \cdot F \, dV = \iint_{\partial\Omega} u F \cdot n \, dA \tag{3}
\]
which is just (2).

c) In the special case of \( F = \nabla u \), the equation (2) is the identity
\[
\iiint_{\Omega} u \nabla \cdot \nabla u \, dV = \iint_{\partial\Omega} u \nabla u \cdot n \, dA - \iiint_{\Omega} |\nabla u|^2 \, dV. \tag{4}
\]
Use this to show that if \( \nabla \cdot \nabla u = 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial\Omega \), then \( u = 0 \) in all of \( \Omega \).

[Remark: \( \nabla \cdot \nabla u = u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} \), the Laplacian, is often written as \( \Delta u \).]

**Solution:**
If \( \nabla \cdot \nabla u = 0 \) in \( \Omega \) and \( u = 0 \) on \( \partial\Omega \), then (4) implies that \( \iiint_{\Omega} |\nabla u|^2 \, dV = 0 \). Hence \( \nabla u = 0 \) in \( \Omega \). Consequently \( u = \text{constant} \). But since \( u = 0 \) on \( \partial\Omega \), then \( u(x) \equiv 0 \) in \( \Omega \).

8. Let \( \vec{x}, \vec{y} \in \mathbb{R}^n \) with the usual inner product which we write as \( \langle x, y \rangle \) (the notation \( \vec{x} \cdot \vec{y} \) is also often used). Also, we write the norm as \( |\vec{x}| = \sqrt{\langle \vec{x}, \vec{x} \rangle} \).

Let \( A \) be a real symmetric \( n \times n \) matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Show that
\[
\langle \vec{x}, A\vec{x} \rangle \geq \lambda_1 |\vec{x}|^2 \quad \text{for all } \vec{x}.
\]

**SOLUTION:**
Version 1. Since \( A \) is a real symmetric matrix, then \( \mathbb{R}^n \) has an orthonormal basis of eigenvectors \( \vec{v}_1, \ldots, \vec{v}_n \) of \( A \), so \( A\vec{v}_j = \lambda_j \vec{v}_j, \ j = 1, \ldots, n \).

Write \( \vec{x} \in \mathbb{R}^n \) in this basis:
\[
\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n.
\]

Using the orthonormality of the \( \vec{v}_j \):
\[
|\vec{x}|^2 = c_1^2 + c_2^2 + \cdots + c_n^2.
\]

Also,
\[
A\vec{x} = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \cdots + c_n \lambda_n \vec{v}_n
\]
so, again using the orthonormality of the \( \vec{v}_j \),
\[
\langle \vec{x}, A\vec{x} \rangle = \lambda_1 c_1^2 + \lambda_2 c_2^2 + \cdots + \lambda_n c_n^2 \\
\geq \lambda_1 (c_1^2 + c_2^2 + \cdots + c_n^2) \\
= \lambda_1 |\vec{x}|^2.
\]

Version 2. This direct proof does not use the characteristic polynomial. Historically, since it concerns the geometric principal axes of the conic \( \langle \vec{x}, A\vec{x} \rangle = 1 \), it probably
predates Version 1, particularly for positive definite matrices. [It also generalizes to give the eigenvalues of the Laplacian.]

The quadratic \( \varphi(\vec{x}) := \langle \vec{x}, A\vec{x} \rangle \) is a continuous function on the unit sphere \( S = \{ \vec{x} \in \mathbb{R}^n \mid |\vec{x}| = 1 \} \). Since \( S \) is compact, there is a point \( \vec{v} \in S \) where \( \varphi(x) \) has its minimum:

\[
\mu := \langle \vec{v}, A\vec{v} \rangle = \min_{\vec{x} \neq 0} \frac{\langle \vec{x}, A\vec{x} \rangle}{|\vec{x}|^2}.
\]

We first show that \( A\vec{v} = \mu \vec{v} \) so \( \mu \) is an eigenvalue of \( A \). Let \( \vec{z} \in \mathbb{R}^n \) and \( \vec{x} = \vec{v} + t\vec{z} \) for small \( t \in \mathbb{R} \) (so \( \vec{x} \neq 0 \)). Let

\[
h(t) := \varphi(\vec{x}) = \frac{\langle \vec{v} + t\vec{z}, A(\vec{v} + t\vec{z}) \rangle}{|\vec{v} + t\vec{z}|^2}.
\]

Then by definition of \( \vec{v} \), \( h(t) \) has its minimum at \( t = 0 \) so \( h(t) \geq h(0) = \mu \). Thus \( h'(0) = 0 \) for all possible choices of \( \vec{z} \). We compute \( h'(0) \) by routine calculus:

\[
h'(0) = \frac{\langle \vec{z}, A\vec{v} \rangle + \langle \vec{v}, A\vec{z} \rangle}{|\vec{v}|^2} - \frac{\langle \vec{v}, A\vec{v} \rangle + 2\langle \vec{v}, \vec{z} \rangle}{|\vec{v}|^4}.
\]

\[
= 2\langle A\vec{v}, \vec{z} \rangle - 2\mu \langle \vec{v}, \vec{z} \rangle - 2\langle A\vec{v} - \mu \vec{v}, \vec{z} \rangle.
\]

Since \( h'(0) = 0 \), then \( \langle A\vec{v} - \mu \vec{v}, \vec{z} \rangle = 0 \) for all vectors \( \vec{z} \in \mathbb{R}^n \). Consequently \( A\vec{v} - \mu \vec{v} = 0 \) so \( \vec{v} \) is an eigenvector with eigenvalue \( \mu \). Note that any eigenvector \( \vec{u} \) of \( A \) with eigenvalue \( \lambda \) satisfies \( \langle \vec{u}, A\vec{u} \rangle = \lambda |\vec{u}|^2 \geq \mu |\vec{u}|^2 \) so \( \mu \) is the smallest eigenvalue.

[In Version 2 to show that \( \vec{v} \) is an eigenvector of \( A \) we could also have used Lagrange multipliers.]

9. Let \( f : \mathbb{R} \to \mathbb{R} \) be an infinitely differentiable function with the properties \( f(0) = 3, \ f(1) = 1, \) and \( f(3) = 5 \). Find an explicit positive real number \( A \) such that there exists a real number \( c \) with \( 0 < c < 3 \) such that \( f''(c) \geq A \).

**10 points**

**SOLUTION:** By the mean value theorem there are points \( 0 < c_1 < 1 \) and \( 1 < c_2 < 3 \) where

\[
f'(c_1) = \frac{1 - 3}{1 - 0} = -2 \quad \text{and} \quad f'(c_2) = \frac{5 - 1}{3 - 1} = 2.
\]

Applying the mean value theorem to the interval \( [c_1, c_2] \) there is a point \( c_1 < c < c_2 \) where

\[
f''(c) = \frac{f''(c_2) - f''(c_1)}{c_2 - c_1} = \frac{2 - (-2)}{c_2 - c_1} > \frac{4}{3}.
\]

We can therefore take \( A = 4/3 \) (or any smaller positive value).

*[Remark: For the optimal value of \( A \) let \( p(x) = \alpha x^2 + \beta x + \gamma \) be the unique quadratic polynomial passing through these three points. Then \( A_{\text{optimal}} = 2\alpha \).]
10. Let $R$ denote the ring $\frac{\mathbb{Z}[x]}{(2x^2 + 2x + 1)}$. Prove that $R$ is an integral domain.

SOLUTION: To show that $R$ is a domain, we need to show that the ideal

$$(2x^2 + 2x + 1)$$

is prime. Since $\mathbb{Z}$ is a UFD, Gauss’s lemma tells us that $\mathbb{Z}[x]$ is a UFD. As a result, it suffices to show that $2x^2 + 2x + 1$ is irreducible in $\mathbb{Z}[x]$. Since $\mathbb{Z}[x] \subset \mathbb{Q}[x]$, it suffices to show $2x^2 + 2x + 1$ is irreducible in $\mathbb{Q}[x]$. Since $2x^2 + 2x + 1$ is of degree 2, this is tantamount to asking whether $2x^2 + 2x + 1$ has a linear factor in $\mathbb{Q}[x]$. Since the discriminant of the polynomial equals $-4$, the quadratic polynomial $2x^2 + 2x + 1$ has no roots in $\mathbb{Q}$. This completes the proof.

11. Find an integer $N$ so that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N} > 100$.

SOLUTION: Use the geometric idea of the integral test:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > \int_1^{N+1} \frac{1}{x} \, dx = \ln(N + 1).$$

Thus pick $\ln(N + 1) > 100$, that is, $N + 1 > e^{100}$; we may take $N$ to be the greatest integer in $e^{100}$.

ALTERNATE A direct grouping of terms. Let $N = 2^k$. Then

$$S_N := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1} + 1} + \cdots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^k} + \cdots + \frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$$

$$= 1 + \frac{k}{2}.$$  

Pick $k$ so that $1 + (k/2) = 100$, that is, $k = 198$. Then $S_{2^{198}} > 100$.

12. Let $A$ be an $n \times n$ real or complex matrix.

a) Show that $\ker A^j \subset \ker A^{j+1}$. If $\ker A^k = \ker A^{k+1}$ for some $k$, show that $\ker A^j = \ker A^k$ for all $j \geq k$.

5 points  

SOLUTION: If $A^j x = 0$ then $A^{j+1} x = A(A^j x) = 0$.

For the second part, by induction, say $x \in \ker A^{k+2}$. Then $0 = A^{k+2} x = A^{k+1}(Ax)$ so $Ax \in \ker A^{k+1}$. But then $Ax \in \ker A^k$, that is, $A^{k+1} x = 0$. 

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b) Say $A$ is a nilpotent $5 \times 5$ matrix. Is it true that $A^5 = 0$? Proof or counterexample.

5 points

SOLUTION (not using part a). Since $A$ is nilpotent, it satisfies $A^k = 0$ for some $k$. The characteristic polynomial, $\lambda^5$, is divisible by the minimal polynomial, $p(\lambda) = \lambda^m$, of $A$. Thus $m \leq 5$, and $A$ satisfies $A^5 = 0$.

ALTERNATE (this approach uses part a). Note that $\dim \ker A^j$ is strictly increasing until it remains constant. Since $\dim \ker A \leq 5$ and $\dim \ker A \geq 1$, it can only increase 4 times so $A^5 = 0$. 