Name: ____

Final Exam for Math 110, Fall 2017

SOLUTIONS

December 14, 2017

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

- You have two hours for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. A cheat sheet is allowed, two-pages, front and back, provided it is freshly handwritten by you.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

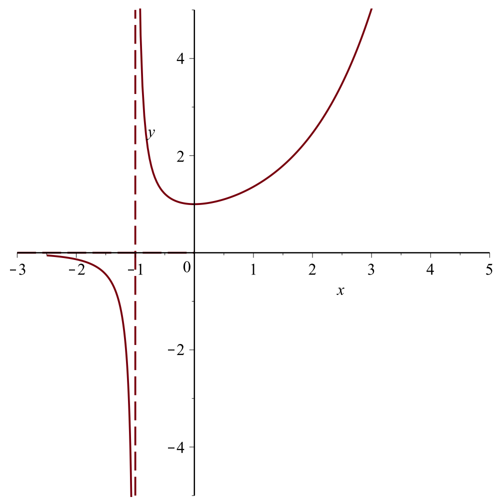
My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. Sketch a graph of $y = e^x/(1+x)$. You do not need to include many exact points but be sure to choose a good scale and to include features such as asymptotes, max/min and discontinuities.

   Solution:



   There is a vertical asymptote at $x = -1$ and a horizontal asymptote at zero as $x \to -\infty$. There is a local minimum at $(0, 1)$.

2. (a) Estimate $\sqrt{9.2}$ using the linearization of the function $\sqrt{x}$ at $x = 9$.

Solution: Let $f(x) = \sqrt{x}$. Then $f'(x) = 1/(2\sqrt{x})$. We have $f(9) = 3$ and $f'(9) = 1/6$, therefore the linearization is $L(x) = 3 + (x-9)/6$. This means

$$L(9.2) = 3 + \frac{0.2}{6} = 3\frac{1}{30} = 3.0333\ldots .$$

(b) Use Taylor's theorem with remainder to give an upper bound on how much this estimate could be off by.

Solution:
The linearization $L(x)$ above is the first degree Taylor polynomial at $a = 9$. Taylor's Theorem with remainder states that

$$f(x) - P_1(x) = \frac{f''(c)}{2}(x-9)^2$$

for some c in the interval $[9, 9.2]$. The value of $f''(x) = -1/(4x^{3/2})$ has its greatest magnitude on this interval at the left endpoint, with $f''(9) = -1/108$, and increases (decreasing in magnitude) as x increases. Therefore

$$-\frac{1/108}{2}(0.2)^2 \leq f(x) - P_1(x) \leq 0$$

which means the error is at most $1/(108 \cdot 2 \cdot 25) = 1/5400 < 0.0002$.

3. "Dollar Bill" Stearn was convicted of tax evasion. As of January, 2017, he owed the U.S. Treasury a total of \$14 million in back taxes. The payment agreement requires him to pay back \$1.5 million each year on January 1, beginning in 2018. Federally mandated interest charges add 10% to his bill every December 31.

   (a) How many millions of dollars does he owe on the morning of January 2 of each of these years?

   Solution:

2017	14 (just checking you understood the setup)
2018	$14 \cdot 1.1 - 1.5 = 13.9$
2019	$(14 \cdot 1.1 - 1.5) \cdot 1.1 - 1.5 = 14 \cdot 1.1^2 - 1.5 \cdot 1.1 - 1.5 = 13.79$
$2017 + n$:	$14 \cdot 1.1^n - 1.5 \sum_{k=0}^{n-1} 1.1^k$

   (b) In approximately what year will his last payment occur? The last payment may be less than the others.

   Solution: This occurs in year $2017{+}n$ where n is large enough to make $14 \cdot 1.1^n$ approximately equal to $1.5 \cdot \sum_{k=0}^{n-1} 1.1^k$. Using the formula for finite geometric sums, we set

$$1.4(1.1)^n = 1.5\left(\frac{1.1^n - 1}{1.1 - 1}\right) = 15(1.1)^n - 15.$$

   Therefore, $15 = (1.1)^n$, hence $n = \log_{1.1} 15 = \log 15/\log 1.1$. Using base 10 logarithms we approximate $\log 15 = log 3 + log 5 \approx 0.477 + 0.7 = 1.177$. By linearization of the log function near 1, because the derivative of $\log x$ at 1 is $1/\ln(10) \approx 1/2.3$, we see that $\log 1.1 \approx (0.1)/2.3$. Therefore,

$$n \approx \frac{1.177 \times 2.3}{0.1} = 11.77 \times 2.3 \approx 27\,.$$

   Thus, Dollar Bill might be done paying in roughly the year 2044.

4. Let $f(x, y) = x/y$.

(a) Compute ∇f.

Solution:

$$\frac{\partial f}{\partial x} = \frac{1}{y} \text{ and } \frac{\partial f}{\partial y} = -\frac{x}{y^2}$$

therefore, $\nabla f = \dfrac{1}{y}\,\hat{\mathbf{i}} - \dfrac{x}{y^2}\,\hat{\mathbf{j}}$.

(b) Evaluate this at the point $(2, 3)$.

$$\nabla f(2, 3) = \frac{1}{3}\,\hat{\mathbf{i}} - \frac{2}{9}\,\hat{\mathbf{j}}$$

(c) Find a unit vector $\mathbf{u}$ parallel to the vector $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$.

Solution: $|3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}| = \sqrt{18} = 3\sqrt{2}$, therefore $u = (3\hat{\mathbf{i}} + 3\hat{\mathbf{j}})\frac{1}{3\sqrt{2}} = \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$.

(d) What is the rate of change of f if you travel from the point $(2, 3)$ in the direction $\mathbf{u}$?

Solution: $\nabla f \cdot \mathbf{u} = \frac{1}{9\sqrt{2}}$.

(e) What is the direction of fastest increase of f at $(2, 3)$?

Solution: This is the direction of the gradient. The length of the gradient is $\sqrt{13}/9$, and dividing by this gives the unit vector $\frac{3}{\sqrt{13}}\hat{\mathbf{i}} - \frac{2}{\sqrt{13}}\hat{\mathbf{j}}$.

(f) How fast does f increase if you travel in this direction?

Solution: This is the length of the gradient vector, which is $\sqrt{13}/9$.

5. An estuary contains 10 kg. of PCB (toxic waste) dispersed in a total of 3 MCF (million cubic feet) of mud. Influx of muddy water will add to the mud volume at the rate of 1 MCF / year. Superfund begins to clean up the site using an enzyme that breaks down (hence removes) the PCB at a rate proportional to the **concentration** in kg/MCF of PCB.

(a) Write an initial value problem for the amount $P(t)$ of PCB present after t years. Be sure to state interpretations and units for any constants you introduce.

Solution: The concentration is the amount of PCB divided by the amount of mud, which is given after t years by $P(t)/(3+t)$. Therefore,

$$P'(t) = -k\frac{P(t)}{3+t}\,.$$

where k is a proportionality constant. The units of the left side are PCB / time. The units of the right, other than k, are PCB/MCF, because 3+t is really 3 MCF + (t years) (1 MCF/year). Therefore k has units of MCF/year. The initial value is $P(0) = 10$.

(b) Solve this IVP to express $P(t)$ as an explicit function of t.
Solution: This is separable, written as $\frac{P'}{P} = \frac{-k}{3+t}$. Integrating,

$$\ln|P(t)| = -k\ln|3+t| + C\,.$$

Both P and t are positive, so we can get rid of the absolute value signs and exponentiate, yielding $P(t) = C(3+t)^{-k}$. The initial value gives us $10 = P(0) = C\cdot 3^{-k}$ therefore $C = 10\cdot 3^k$ and the particular solution is

$$P(t) = 10\left(\frac{3}{3+t}\right)^k\,.$$

6. (a) Write the improper integral $\int_0^1 x^2 \ln(x)\, dx$ as a limit of ordinary integrals.

Solution:

$$\int_0^1 x^2 \ln(x)\, dx = \lim_{b \to 0^+} \int_b^1 x^2 \ln(x)\, dx \,.$$

(b) Evaluate the ordinary integrals.

Solution: Integrating by parts,

$$\begin{aligned} \int_b^1 x^2 \ln(x)\, dx &= \frac{1}{3}x^3 \ln x \Big|_b^1 - \int_b^1 \frac{1}{3}x^2 \\ &= -\frac{1}{3}b^3 \ln b + \frac{1}{9}(1 - b^3) \,. \end{aligned}$$

(c) Evaluate the limit, justifying as necessary.

Solution: Using L'Hôpital's rule on $\frac{\ln b}{b^{-3}}$ or else using the fact that $|\ln b| \ll b^{-3}$ as $b \to 0^+$, we see that the first of the two terms has a limit of 0 at 0. Therefore the limit evaluates to 1/9.

7. (a) Solve the equation $\frac{dy}{dx} - \frac{1}{x}y = e^{-x}$ with initial value $y(1) = 3$. It is OK if part of the answer must be left as an integral; be precise about the limits of integration.

Solution: This is in the form $y' + P(x)y = Q(x)$ with $P(x) = -1/x$ and $Q(x) = e^{-x}$. Letting $v(x) = e^{\int P(x)\,dx}$ be the integrating factor, we compute the anitderivative of $P(x)$ as $-\ln|x|$, hence $v(x) = 1/|x|$. We are concerned with values of y for $x \geq 1$, hence $x > 0$ and we can get rid of the absolute value signs: $v(x) = 1/x$. Multiplying through by the integrating factor gives

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{e^{-x}}{x}$$

and integrating gives

$$\frac{1}{x}y = \int \frac{e^{-x}}{x}\,dx\,.$$

rewriting as a definite integral with x as an upper limit and a different bound variable gives

$$y = x\left(\int_1^x \frac{e^{-t}}{t}\,dt + C\right)$$

and using the initial value gives $3 = y(1) = 1 \cdot (0 + C)$ therefore $C = 3$ and the solution is

$$y(x) = 3x + x\int_1^x \frac{e^{-t}}{t}\,dt\,.$$

(b) Estimate the value $y(3)$ by using a Riemann sum with two rectangles to estimate the integral.

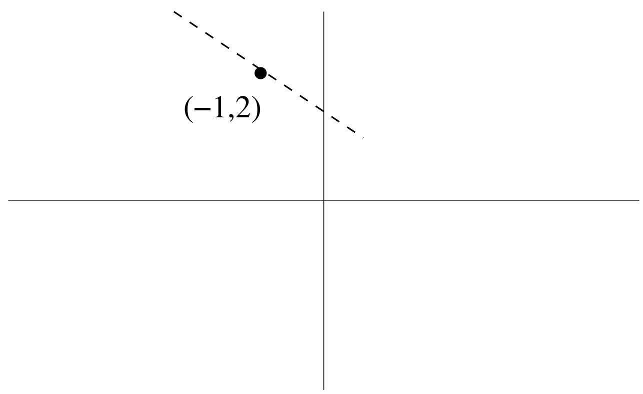
Solution: $y(3) = 9 + 3\int_1^3 (e^{-t}/t)\,dt$. Breaking the interval $[1, 3]$ into two parts and using a left Riemann sum gives the approximation $e^{-1}/1 + e^{-2}/2$ for the integral. A right Riemann sum, midpoint Riemann sum or trapezoidal approximation would also work, perhaps even more accurately. The result is $9 + 3e^{-1} + (3/2)e^{-2} \approx 9 + 1.1 + 0.2 = 10.3$. Note: the left Riemann sum is a significant overestimate.

8. Suppose f is a function of x and y, and that you know

$$\begin{aligned}\frac{\partial f}{\partial x}(-1,2) &= 2\,;\\ \frac{\partial f}{\partial y}(-1,2) &= 3\,.\end{aligned}$$

(a) Sketch a piece of the level set of f that passes through the point $(-1,2)$.

Solution: Level sets of f are perpendicular to the gradient of f, therefore the piece near $(-1,2)$ is nearly a line segment with slope $-2/3$.



(b) If the x-axis represents **rating** on a scale of -5 to 5 and the y-axis represents **efficiency** on a scale of 0 to 10, what is the marginal rate of substitution of efficiency for rating when staying on the level set of f through the point $(-1,2)$?

Solution: The marginal rate of substitution is $-dy/dx = (\partial f/\partial x)/(\partial f/\partial y)$ which is equal to $2/3$ at the point $(-1,2)$.

(c) Suppose $x(t) = t^2 - 5$ and $y(t) = \dfrac{4}{t}$. What is $\dfrac{df}{dt}(2)$?

Solution: By the multivariate chain rule, $df/dt = \partial f/\partial x)(dx/dt) + (\partial f/\partial y)(dy/dt)$. When $t = 2$, the point $(x,y) = (-1,2)$. At the point $(-1,2)$,

$$\frac{df}{dt} = 2x' + 3y' = (4t - 3\cdot 4/t^2)|_{t=2} = 8 - 3 = 5\,.$$

9. In (a)–(c) find constants c and p such that $f(x) \sim cx^p$ as $x \to \infty$.

(a) $f(x) = x \cdot (3x^2 + 2x\ln(x) + 1)$
Solution: $f(x) \sim 3x^3$ because both $x \ln x$ and 1 are much smaller than x^2 as $x \to \infty$.

(b) $f(x) = \dfrac{\sqrt{e^{2x}+1}}{xe^x}$.
Solution: Use the fact that $1 \ll e^{2x}$ to see that

$$f(x) \sim \sqrt{e^{2x}}/(xe^x) = x^{-1}\,.$$

(c) EXTRA CREDIT: $f(x) = (x+1)^{1/3} - x$.
Solution: Using the linearization near $t = x$, for the function $t^{1/3}$ we see that $(x+1)^{1/3} \approx L(x+1) = x^{1/3} + (1/3)x^{-2/3}$. Therefore $f(x) \approx (1/3)x^{-2/3}$ and the answer is $c = 1/3, p = -2/3$.

(d) Does $\displaystyle\int_1^\infty \frac{\sqrt{e^{2x}+1}}{xe^x}\,dx$ converge or diverge? Please give a reason.
Solution: By part (b), $f(x) \sim x^{-1}$, so the integral of $f(x)$ converges if and only if the integral of x^{-1} converges. By the p-test, $\int_b^\infty x^p\,dx$ converges if and only if $p < -1$, which fails when $p = -1$, hence the integral is divergent.

10. Microsoft has 1 gigabuck to finance its new communication technology. It plans to spend x gigabucks for signal power, and to develop the product for y decades before charging for it, which will cost y gigabucks. The value of the product is estimated to be proportional to the square of the signal strength (signal strength is proportional to x), to e^{ky} (due to exponential growth of the market share during the free period) and to $1 - x - y$ (cash remaining provides liquidity of the communication division).

   (a) Write down a formula for the value V of this project. Explain all constants and variables other than x and y and give units where appropriate.

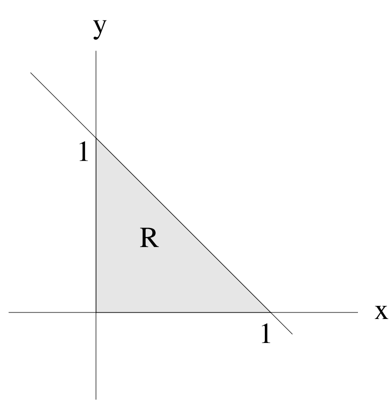
   Solution:

$$V = C(bx^2)e^{ky}(1 - x - y)$$

   where V is value in gigabucks, k is in units of inverse years, so e^{ky} is unitless, x and y are in gigabucks (y is decades times 1 gigabuck per decade), b has units of signal strength per square gigabuck, and C has units of inverse signal strength. Alternatively, you could combine b and C into one constant C' that has units of gb^{-2}.

   (b) The choice of x and y that produces the greatest project value the function over what region R?

   Solution: The region R for which the problem makes sense is $x \geq 0, y \geq 0$ and $x + y \leq 1$.



(c) Solve the optimization problem when $k = 4$. Three quarters of the credit will be given if you set up the problem correctly.

Solution: We can ignore the constant and maximize $V_* = x^2 e^{ky}(1-x-y)$ over R. The maximum value could be interior to R or on one of the three boundary segments. At an interior maximum, ∇R has tovanish, that is, both $\partial V/\partial x$ and $\partial V/\partial y$ have to be zero. We compute:

$$\begin{aligned}\frac{\partial V_*}{\partial x} &= e^{ky}(2x(1-x-y)-x^2)\\ \frac{\partial V_*}{\partial y} &= e^{ky}x^2(k(1-x-y)-1)\,.\end{aligned}$$

In the interior x is not zero. Setting both partial derivatives equal to zero, we may factor out x from both, leading to

$$\begin{aligned}2-3x-3y &= 0\\ (k-1)-kx-ky &= 0\,.\end{aligned}$$

When $k = 4$, the solution is $x = 1/2, y = 1/4$. This is a candidate for the maximum. The value here is $(1/2)^2 e^1(1-1/2-1/4) = e/16 \approx 2.7/16 = 0.16875$.

The value is zero when x or $1-x-y$ is zero, therefore the only other candidates for maximum are along the x axis where $y = 0$ and $0 \le x \le 1$. There, $V = x^2(1-x)$ which has a maximum of $4/27 \approx 0.148\ldots$ at $x = 2/3$. The interior solution does better.

The solution is therefore $\boxed{\text{V(1/2,1/4) = e/16}}$.

(d) Solve the optimization problem when $k = 2$.

Solution: This time we have $1-2x-2y = 0$ instead of $3-4x-4y = 0$, consequently the solution is $x = 0, y = 1$. This is not in the interior, therefore the maximum is on the boundary. We have already seen this has to be at $(2/3, 0)$ where the value is $4/27 \approx 0.148\ldots$.

The solution is therefore $\boxed{\text{V(2/3,0) = 4/27}}$.

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$
\begin{aligned}
e &\approx 2.7 \\
\ln(2) &\approx 0.7 \\
\ln(10) &\approx 2.3 \\
\log_{10}(2) &\approx 0.3 \\
\log_{10}(3) &\approx 0.48
\end{aligned}
$$

Some other useful quantities to with 1%:

$$
\begin{aligned}
\pi &\approx \frac{22}{7} \\
\sqrt{10} &\approx \pi \\
\sqrt{2} &\approx 1.4 \\
\sqrt{1/2} &\approx 0.7
\end{aligned}
$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

$$\int \frac{e^x}{x^n}\,dx \text{ cannot be simplified}$$

$$\int e^{x^a}\,dx \text{ cannot be simplified unless } a = 1/n \text{ for some integer } n$$

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \frac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$