Final Exam for Math 110, Fall 2017

SOLUTIONS

December 14, 2017

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- You have two hours for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn’t have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. A cheat sheet is allowed, two-pages, front and back, provided it is freshly handwritten by you.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this midterm examination.

Signature: ________________________________

Date: ________________________________
1. Sketch a graph of $y = e^x/(1 + x)$. You do not need to include many exact points but be sure to choose a good scale and to include features such as asymptotes, max/min and discontinuities.

**Solution:**

There is a vertical asymptote at $x = -1$ and a horizontal asymptote at zero as $x \to -\infty$. There is a local minimum at $(0, 1)$.
2. (a) Estimate $\sqrt{9.2}$ using the linearization of the function $\sqrt{x}$ at $x = 9$.

**Solution:** Let $f(x) = \sqrt{x}$. Then $f'(x) = 1/(2\sqrt{x})$. We have $f(9) = 3$ and $f'(9) = 1/6$, therefore the linearization is $L(x) = 3 + (x - 9)/6$. This means

$$L(9.2) = 3 + \frac{0.2}{6} = 3 \frac{1}{30} = 3.0333\ldots.$$ 

(b) Use Taylor’s theorem with remainder to give an upper bound on how much this estimate could be off by.

**Solution:**

The linearization $L(x)$ above is the first degree Taylor polynomial at $a = 9$. Taylor’s Theorem with remainder states that

$$f(x) - P_1(x) = \frac{f''(c)}{2}(x - 9)^2$$

for some $c$ in the interval [9, 9.2]. The value of $f''(x) = -1/(4x^{3/2})$ has its greatest magnitude on this interval at the left endpoint, with $f''(9) = -1/108$, and increases (decreasing in magnitude) as $x$ increases. Therefore

$$-\frac{1/108}{2}(0.2)^2 \leq f(x) - P_1(x) \leq 0$$

which means the error is at most $1/(108 \cdot 2 \cdot 25) = 1/5400 < 0.0002$. 

3
3. “Dollar Bill” Stearn was convicted of tax evasion. As of January, 2017, he owed the U.S. Treasury a total of $14 million in back taxes. The payment agreement requires him to pay back $1.5 million each year on January 1, beginning in 2018. Federally mandated interest charges add 10% to his bill every December 31.

(a) How many millions of dollars does he owe on the morning of January 2 of each of these years?

Solution:

\[
\begin{align*}
2017 & : 14 \\
2018 & : 14 \cdot 1.1 - 1.5 = 13.9 \\
2019 & : (14 \cdot 1.1 - 1.5) \cdot 1.1 - 1.5 = 14 \cdot 1.1^2 - 1.5 \cdot 1.1 - 1.5 = 13.79 \\
2017 + n : & \quad 14 \cdot 1.1^n - 1.5 \sum_{k=0}^{n-1} 1.1^k
\end{align*}
\]

(b) In approximately what year will his last payment occur? The last payment may be less than the others.

Solution: This occurs in year \(2017+n\) where \(n\) is large enough to make \(14 \cdot 1.1^n\) approximately equal to \(1.5 \cdot \sum_{k=0}^{n-1} 1.1^k\). Using the formula for finite geometric sums, we set

\[
1.4(1.1)^n = 1.5 \left( \frac{1.1^n - 1}{1.1 - 1} \right) = 15(1.1)^n - 15.
\]

Therefore, \(15 = (1.1)^n\), hence \(n = \log_{1.1} 15 = \log 15 / \log 1.1\). Using base 10 logarithms we approximate \(\log 15 = \log 3 + \log 5 \approx 0.477 + 0.7 = 1.177\). By linearization of the log function near 1, because the derivative of \(\log x\) at 1 is \(1/\ln(10) \approx 1/2.3\), we see that \(\log 1.1 \approx (0.1)/2.3\). Therefore,

\[
n \approx \frac{1.177 \times 2.3}{0.1} = 11.77 \times 2.3 \approx 27.
\]

Thus, Dollar Bill might be done paying in roughly the year 2044.
4. Let \( f(x, y) = x/y \).

(a) Compute \( \nabla f \).

**Solution:**

\[
\frac{\partial f}{\partial x} = \frac{1}{y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}
\]

therefore, \( \nabla f = \frac{1}{y} \mathbf{i} - \frac{x}{y^2} \mathbf{j} \).

(b) Evaluate this at the point \((2, 3)\).

\[
\nabla f(2, 3) = \frac{1}{3} \mathbf{i} - \frac{2}{9} \mathbf{j}
\]

(c) Find a unit vector \( \mathbf{u} \) parallel to the vector \( 3\mathbf{i} + 3\mathbf{j} \).

**Solution:**

\[
|3\mathbf{i} + 3\mathbf{j}| = \sqrt{18} = 3\sqrt{2}, \quad \text{therefore} \quad u = (3\mathbf{i} + 3\mathbf{j}) \frac{1}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j}.
\]

(d) What is the rate of change of \( f \) if you travel from the point \((2, 3)\) in the direction \( \mathbf{u} \)?

**Solution:**

\[ \nabla f \cdot \mathbf{u} = \frac{1}{9\sqrt{2}} \]

(e) What is the direction of fastest increase of \( f \) at \((2, 3)\)?

**Solution:** This is the direction of the gradient. The length of the gradient is \( \sqrt{13}/9 \), and dividing by this gives the unit vector \( \frac{3}{\sqrt{13}} \mathbf{i} - \frac{2}{\sqrt{13}} \mathbf{j} \).

(f) How fast does \( f \) increase if you travel in this direction?

**Solution:** This is the length of the gradient vector, which is \( \sqrt{13}/9 \).
5. An estuary contains 10 kg. of PCB (toxic waste) dispersed in a total of 3 MCF (million cubic feet) of mud. Influx of muddy water will add to the mud volume at the rate of 1 MCF / year. Superfund begins to clean up the site using an enzyme that breaks down (hence removes) the PCB at a rate proportional to the concentration in kg/MCF of PCB.

(a) Write an initial value problem for the amount $P(t)$ of PCB present after $t$ years. Be sure to state interpretations and units for any constants you introduce.

**Solution:** The concentration is the amount of PCB divided by the amount of mud, which is given after $t$ years by $P(t)/(3+t)$. Therefore,

$$P'(t) = -k \frac{P(t)}{3 + t}.$$ 

where $k$ is a proportionality constant. The units of the left side are PCB / time. The units of the right, other than $k$, are PCB/MCF, because $3+t$ is really $3$ MCF + (t years) (1 MCF/year). Therefore $k$ has units of MCF/year. The initial value is $P(0) = 10$.

(b) Solve this IVP to express $P(t)$ as an explicit function of $t$.

**Solution:** This is separable, written as $rac{P'}{P} = -\frac{k}{3+t}$. Integrating,

$$\ln |P(t)| = -k \ln |3 + t| + C .$$

Both $P$ and $t$ are positive, so we can get rid of the absolute value signs and exponentiate, yielding $P(t) = C(3 + t)^{-k}$. The initial value gives us $10 = P(0) = C \cdot 3^{-k}$ therefore $C = 10 \cdot 3^k$ and the particular solution is

$$P(t) = 10 \left( \frac{3}{3 + t} \right)^k .$$
6. (a) Write the improper integral \( \int_0^1 x^2 \ln(x) \, dx \) as a limit of ordinary integrals.

\[
\int_0^1 x^2 \ln(x) \, dx = \lim_{b \to 0^+} \int_b^1 x^2 \ln(x) \, dx.
\]

(b) Evaluate the ordinary integrals.

\[
\text{Solution: Integrating by parts,} \\
\int_b^1 x^2 \ln(x) \, dx = \frac{1}{3} x^3 \ln x \bigg|_b^1 - \int_b^1 \frac{1}{3} x^2 \\
= -\frac{1}{3} b^3 \ln b + \frac{1}{9} (1 - b^3).
\]

(c) Evaluate the limit, justifying as necessary.

\[
\text{Solution: Using L'Hôpital's rule on } \frac{\ln b}{b^{-3}} \text{ or else using the fact that} \\
|\ln b| \ll b^{-3} \text{ as } b \to 0^+, \text{ we see that the first of the two terms has a} \\
\text{limit of 0 at 0. Therefore the limit evaluates to } \frac{1}{9}.
\]
7. (a) Solve the equation \( \frac{dy}{dx} - \frac{1}{x}y = e^{-x} \) with initial value \( y(1) = 3 \). It is OK if part of the answer must be left as an integral; be precise about the limits of integration.

**Solution:** This is in the form \( y' + P(x)y = Q(x) \) with \( P(x) = -1/x \) and \( Q(x) = e^{-x} \). Letting \( v(x) = e^{\int P(x)dx} \) be the integrating factor, we compute the antiderivative of \( P(x) \) as \(-\ln|x|\), hence \( v(x) = 1/|x| \). We are concerned with values of \( y \) for \( x \geq 1 \), hence \( x > 0 \) and we can get rid of the absolute value signs: \( v(x) = 1/x \). Multiplying through by the integrating factor gives

\[
\frac{1}{x}y' - \frac{1}{x^2}y = \frac{e^{-x}}{x}
\]

and integrating gives

\[
\frac{1}{x}y = \int \frac{e^{-x}}{x} \, dx.
\]

rewriting as a definite integral with \( x \) as an upper limit and a different bound variable gives

\[
y = x \left( \int_{1}^{x} \frac{e^{-t}}{t} \, dt + C \right)
\]

and using the initial value gives \( 3 = y(1) = 1 \cdot (0 + C) \) therefore \( C = 3 \) and the solution is

\[
y(x) = 3x + x \int_{1}^{x} \frac{e^{-t}}{t} \, dt.
\]

(b) Estimate the value \( y(3) \) by using a Riemann sum with two rectangles to estimate the integral.

**Solution:** \( y(3) = 9 + 3 \int_{1}^{3} \frac{e^{-t}}{t} \, dt \). Breaking the interval \([1, 3]\) into two parts and using a left Riemann sum gives the approximation \( e^{-1}/1 + e^{-2}/2 \) for the integral. A right Riemann sum, midpoint Riemann sum or trapezoidal approximation would also work, perhaps even more accurately. The result is \( 9 + 3e^{-1} + (3/2)e^{-2} \approx 9 + 1.1 + 0.2 = 10.3 \). Note: the left Riemann sum is a significant overestimate.
8. Suppose $f$ is a function of $x$ and $y$, and that you know

$$\frac{\partial f}{\partial x}(-1, 2) = 2;$$

$$\frac{\partial f}{\partial y}(-1, 2) = 3.$$

(a) Sketch a piece of the level set of $f$ that passes through the point $(-1, 2)$.

**Solution:** Level sets of $f$ are perpendicular to the gradient of $f$, therefore the piece near $(-1, 2)$ is nearly a line segment with slope $-2/3$.

![Diagram](image)

(b) If the $x$-axis represents rating on a scale of -5 to 5 and the $y$-axis represents efficiency on a scale of 0 to 10, what is the marginal rate of substitution of efficiency for rating when staying on the level set of $f$ through the point $(-1, 2)$?

**Solution:** The marginal rate of substitution is $-dy/dx = (\partial f / \partial x)/(\partial f / \partial y)$ which is equal to $2/3$ at the point $(-1, 2)$.

(c) Suppose $x(t) = t^2 - 5$ and $y(t) = \frac{4}{t}$. What is $\frac{df}{dt}(2)$?

**Solution:** By the multivariate chain rule, $df/dt = \partial f / \partial x(dx/dt) + (\partial f / \partial y)(dy/dt)$. When $t = 2$, the point $(x, y) = (-1, 2)$. At the point $(-1, 2)$,

$$\frac{df}{dt} = 2x' + 3y' = (4t - 3 \cdot 4/t^2)|_{t=2} = 8 - 3 = 5.$$
9. In (a)–(c) find constants $c$ and $p$ such that $f(x) \sim cx^p$ as $x \to \infty$.

(a) $f(x) = x \cdot (3x^2 + 2x \ln(x) + 1)$

**Solution:** $f(x) \sim 3x^3$ because both $x \ln x$ and 1 are much smaller than $x^2$ as $x \to \infty$.

(b) $f(x) = \frac{\sqrt{e^{2x} + 1}}{xe^x}$.

**Solution:** Use the fact that $1 \ll e^{2x}$ to see that

$$f(x) \sim \frac{\sqrt{e^{2x}}}{(xe^x)} = x^{-1}.$$  

(c) EXTRA CREDIT: $f(x) = (x + 1)^{1/3} - x$.

**Solution:** Using the linearization near $t = x$, for the function $t^{1/3}$ we see that $(x + 1)^{1/3} \approx L(x + 1) = x^{1/3} + (1/3)x^{-2/3}$. Therefore $f(x) \approx (1/3)x^{-2/3}$ and the answer is $c = 1/3, p = -2/3$.

(d) Does $\int_1^\infty \frac{\sqrt{e^{2x} + 1}}{xe^x} \, dx$ converge or diverge? Please give a reason.

**Solution:** By part (b), $f(x) \sim x^{-1}$, so the integral of $f(x)$ converges if and only if the integral of $x^{-1}$ converges. By the $p$-test, $\int_b^\infty x^p \, dx$ converges if and only if $p < -1$, which fails when $p = -1$, hence the integral is divergent.
10. Microsoft has 1 gigabuck to finance its new communication technology. It plans to spend $x$ gigabucks for signal power, and to develop the product for $y$ decades before charging for it, which will cost $y$ gigabucks. The value of the product is estimated to be proportional to the square of the signal strength (signal strength is proportional to $x$), to $e^{ky}$ (due to exponential growth of the market share during the free period) and to $1 - x - y$ (cash remaining provides liquidity of the communication division).

(a) Write down a formula for the value $V$ of this project. Explain all constants and variables other than $x$ and $y$ and give units where appropriate.

**Solution:**

$$V = C(bx^2)e^{ky}(1 - x - y)$$

where $V$ is value in gigabucks, $k$ is in units of inverse years, so $e^{ky}$ is unitless, $x$ and $y$ are in gigabucks ($y$ is decades times 1 gigabuck per decade), $b$ has units of signal strength per square gigabuck, and $C$ has units of inverse signal strength. Alternatively, you could combine $b$ and $C$ into one constant $C'$ that has units of gb$^{-2}$.

(b) The choice of $x$ and $y$ that produces the greatest project value the function over what region $R$?

**Solution:** The region $R$ for which the problem makes sense is $x \geq 0, y \geq 0$ and $x + y \leq 1$. 

![Diagram of the region R](image)
(c) Solve the optimization problem when $k = 4$. Three quarters of the credit will be given if you set up the problem correctly.

**Solution:** We can ignore the constant and maximize $V_* = x^2 e^{ky}(1-x-y)$ over $R$. The maximum value could be interior to $R$ or on one of the three boundary segments. At an interior maximum, $\nabla R$ has to vanish, that is, both $\partial V/\partial x$ and $\partial V/\partial y$ have to be zero. We compute:

$$\frac{\partial V_*}{\partial x} = e^{ky}(2x(1-x-y) - x^2)$$

$$\frac{\partial V_*}{\partial y} = e^{ky}x^2(k(1-x-y) - 1).$$

In the interior $x$ is not zero. Setting both partial derivatives equal to zero, we may factor out $x$ from both, leading to

$$2 - 3x - 3y = 0$$

$$(k - 1) - kx - ky = 0.$$ 

When $k = 4$, the solution is $x = 1/2, y = 1/4$. This is a candidate for the maximum. The value here is $(1/2)^2 e^1(1 - 1/2 - 1/4) = e/16 \approx 2.7/16 = 0.16875$.

The value is zero when $x$ or $1 - x - y$ is zero, therefore the only other candidates for maximum are along the $x$ axis where $y = 0$ and $0 \leq x \leq 1$. There, $V = x^2(1-x)$ which has a maximum of $4/27 \approx 0.148\ldots$ at $x = 2/3$. The interior solution does better.

The solution is therefore $V(1/2,1/4) = e/16$.

(d) Solve the optimization problem when $k = 2$.

**Solution:** This time we have $1-2x-2y = 0$ instead of $3-4x-4y = 0$, consequently the solution is $x = 0, y = 1$. This is not in the interior, therefore the maximum is on the boundary. We have already seen this has to be at $(2/3,0)$ where the value is $4/27 \approx 0.148\ldots$.

The solution is therefore $V(2/3,0) = 4/27$. 

Logarithm Cheat Sheet

These values are accurate to within 1%:

\[
\begin{align*}
  e & \approx 2.7 \\
  \ln(2) & \approx 0.7 \\
  \ln(10) & \approx 2.3 \\
  \log_{10}(2) & \approx 0.3 \\
  \log_{10}(3) & \approx 0.48
\end{align*}
\]

Some other useful quantities to within 1%:

\[
\begin{align*}
  \pi & \approx \frac{22}{7} \\
  \sqrt{10} & \approx \pi \\
  \sqrt{2} & \approx 1.4 \\
  \sqrt{1/2} & \approx 0.7
\end{align*}
\]

(ok so technically \(\sqrt{2}\) is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than \(\sqrt{1/2}\))
\[
\int \frac{e^x}{x^n} \, dx \text{ cannot be simplified}
\]

\[
\int e^{ax} \, dx \text{ cannot be simplified unless } a = 1/n \text{ for some integer } n
\]