Name: ______________________________ Section (circle one): 001 002

Final Exam for Math 110, Fall 2014

December 12, 2014

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	20	
6	12	
7	12	
8	12	
9	12	
10	20	
11	12	
12	6	
13	6	
Total	160	

- You have two hours for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. A cheat sheet of any length is allowed, provided it is freshly handwritten by you.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________

Date: ______________________

1. Graph these functions. The main thing is to get the general shape, but please also mark a few points and any maxima, minima, asymptotes or discontinuities.

   (a) $y = \dfrac{e^x}{x}$

   (b) $y = \dfrac{1}{x^3 - x}$

2. Compute is the trapezoidal approximation to $\int_1^{3/2} \frac{e^x}{x}\, dx$ using just one trapezoid. Leave the expression in exact form here.

Then select which of these is closest to the value you have written down.

(i) 0.3

(ii) 0.5

(iii) 1.0

(iv) 1.4

(v) 2.0

3. (a) Use linearization to estimate $\log_{10} 1.069$.

(b) Use this to estimate $\log_{10}\left(1.069^{30}\right)$.

(c) Use this to estimate 1.069^{30}.

4. Give the general solution of the differential equation $(1+t)y' + y = \sqrt{t}$.

5. Uncle Sam has a debt of 18 trillion dollars (as of January 1, 2015). Senator Paul decides he needs to pay it off, and proposes an installment scheme, under which Uncle Sam will make payments of one trillion dollars on December 31, 2015 and every December 31 thereafter. Unfortunately, the debt increases by 5% during the course of every year.

   (a) Write expressions for the amount owed by Uncle Sam on January 1 of 2015, 2016 and 2017.

   (b) Write an expression, which will involve the Sigma notation, for how much Uncle Sam owes on January 1 of year n, counting 2015 as year 0.

(c) Evaluate the sum to get an algebraic expression.

(d) Solve for the number of years, n, in which the debt will be paid off. Please leave this as an exact expression (it is OK if it leads to a value which is not a whole number).

(e) Estimate the numerical value of n to the nearest whole number.

6. Let $f(x) = \int_1^x \frac{e^x}{x}\, dx$.

(a) Compute the linear and quadratic Taylor polynomials for f about the point $x = 1$.

$$\begin{aligned} L(x) &= \\ P_2(x) &= \end{aligned}$$

(b) Use these to estimate $f(3/2)$. Leave as exact expresssions; do not evaluate numerically.

(c) Is the linear estimate an over- or under-estimate of the true value of $f(3/2)$?

7. Sketch the region and evaluate the integral.

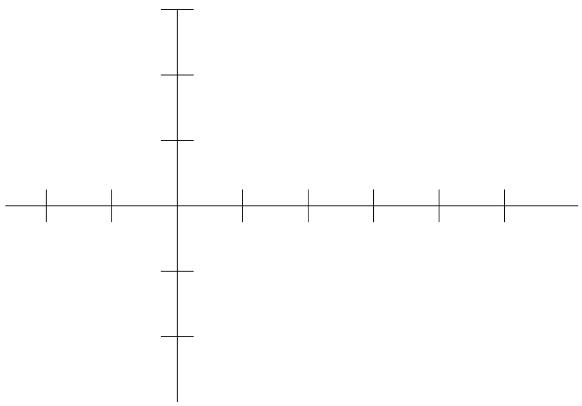
$$\int_4^9 \int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy\, dx$$

8. The price of a turkey is proportional to its weight and inversely proportional to the square of its age. Jack's mother gives him enough money to buy a 10 pound turkey that is one year old. When Jack gets to the fair, he realizes that he needs a turkey that is slightly bigger. How much older will the turkey have to be per extra pound in weight?

   Please begin by writing down an equation satisfied by price, weight and age, giving the interpretation and units for all varialbles and constants used.

9. (a) Compute the gradient of the function $f(x, y) = \dfrac{x^2}{8\sqrt{y}}$.

(b) Evaluate the gradient of f at the point $(4, 1)$ and draw this vector on these coordinate axes starting at the point $(4, 1)$.



(c) From the point $(4, 1)$, which direction should you move in order to increase f the fastest. State this direction by giving a unit vector pointing in the direction.

(d) At the point $(4, 1)$, how fast does f increase per unit moved in this direction?

10. Uncle Sam owes 18 trillion dollars (this is a new problem, really!). Interest accmulates continuously at the rate of 5%/year. Suppose that Congress forces Uncle Sam to pay back the debt continuously at a rate of one trillion dollars per year.

    (a) Write a differential equation for the amount owed by Uncle Sam at time t. Please give the meaning and units of all variables.

    (b) Find the general solution of this differential equation.

(c) State the initial condition and give the solution to the initial value problem.

(d) How long will it take for Uncle Sam's debt to reach zero?

11. (a) Compute the indefinite integral $\int x^{-3} \ln x \, dx$.

(b) Write the improper integral $\int_1^{\infty} x^{-3} \ln x \, dx$ as a limit and evaluate it.

12. Suppose that $f(x,y)$ is a function and that x and y depend on parameters s and t by the formulas $x = \sqrt{s^2+t}$ and $y = \ln(t-4)$. Which of the following expressions correctly describes the rate of change of f with respect to t when (s,t) starts at the value $(2,5)$ and then t is varied while s is held constant? You need only circle the correct number from (i) to (v).

(i) $\frac{2}{3}\frac{\partial f}{\partial x}(2,5) + \frac{\partial f}{\partial y}(2,5)$

(ii) $\frac{2}{3}\frac{\partial f}{\partial x}(3,0) + \frac{\partial f}{\partial y}(3,0)$

(iii) $\frac{1}{6}\frac{\partial f}{\partial x}(2,5) + \frac{\partial f}{\partial y}(2,5)$

(iv) $\frac{1}{6}\frac{\partial f}{\partial x}(3,0) + \frac{\partial f}{\partial y}(3,0)$

(v) None of the above

13. A swimming pool holding 300 cubic meters of water is determined to contain 1/100 of a cubic meter of toxins. Immediately a drain is opened and pool water starts flowing out at 3 cubic meters per minute. Also a hose is inserted to pump in fresh water at the rate of 1 cubic meter per minute. Assuming the fresh water mixes rapidly with the pool water, which of these differential equations models the total amount P of poison in the pool at time t minutes after the toxicity is discovered? You need only circle the correct number from (i) to (v).

(i) $P'(t) = -\dfrac{3P(t)}{300-2t}$

(ii) $P'(t) = -1 + \dfrac{P(t)}{3}$

(iii) $\dfrac{P'(t)}{P(t)} = -\dfrac{1/100}{300}$

(iv) $P'(t) - P(t) = -\dfrac{2}{300-t}$

(v) None of the above

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \\ \log_{10}(3) &\approx 0.48 \end{aligned}$$

Some other useful quantities to with 1%:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \end{aligned}$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (a >$

22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x >$