Name: ____

Final Exam for Math 110, Fall 2017

December 14, 2017

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

- You have two hours for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. A cheat sheet is allowed, two-pages, front and back, provided it is freshly handwritten by you.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ________________________________

Date: ________________________________

1. Sketch a graph of $y = e^x/(1+x)$. You do not need to include many exact points but be sure to choose a good scale and to include features such as asymptotes, max/min and discontinuities.

2. (a) Estimate $\sqrt{9.2}$ using the linearization of the function $\sqrt{x}$ at $x = 9$.

(b) Use Taylor's theorem with remainder to give an upper bound on how much this estimate could be off by.

3. "Dollar Bill" Stearn was convicted of tax evasion. As of January, 2017, he owed the U.S. Treasury a total of \$14 million in back taxes. The payment agreement requires him to pay back \$1.5 million each year on January 1, beginning in 2018. Federally mandated interest charges add 10% to his bill every December 31.

   (a) How many millions of dollars does he owe on the morning of January 2 of each of these years?

   2017 ________________________

   2018 ________________________

   2019 ________________________

   $2017 + n$:

   (b) In approximately what year will his last payment occur? The last payment may be less than the others.

4. Let $f(x, y) = x/y$.

(a) Compute ∇f.

(b) Evaluate this at the point $(2, 3)$.

(c) Find a unit vector $\mathbf{u}$ parallel to the vector $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$.

(d) What is the rate of change of f if you travel from the point $(2, 3)$ in the direction $\mathbf{u}$?

(e) What is the direction of fastest increase of f at $(2, 3)$?

(f) How fast does f increase if you travel in this direction?

5. An estuary contains 10 kg. of PCB (toxic waste) dispersed in a total of 3 MCF (million cubic feet) of mud. Influx of muddy water will add to the mud volume at the rate of 1 MCF / year. Superfund begins to clean up the site using an enzyme that breaks down (hence removes) the PCB at a rate proportional to the **concentration** in kg/MCF of PCB.

   (a) Write an initial value problem for the amount $P(t)$ of PCB present after t years. Be sure to state interpretations and units for any constants you introduce.

   (b) Solve this IVP to express $P(t)$ as an explicit function of t.

6. (a) Write the improper integral $\int_0^1 x^2 \ln(x)\, dx$ as a limit of ordinary integrals.

(b) Evaluate the ordinary integrals.

(c) Evaluate the limit, justifying as necessary.

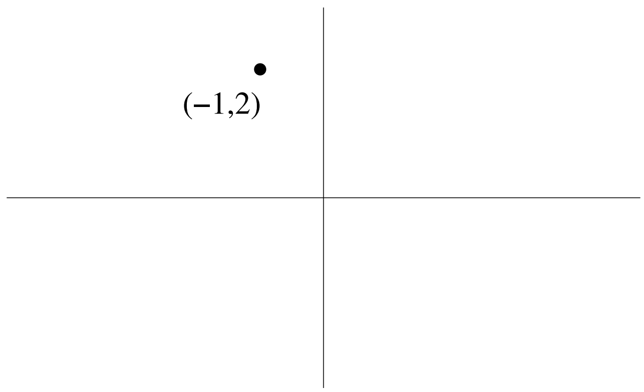
7. (a) Solve the equation $\frac{dy}{dx} - \frac{1}{x}y = e^{-x}$ with initial value $y(1) = 3$. It is OK if part of the answer must be left as an integral; be precise about the limits of integration.

(b) Estimate the value $y(3)$ by using a Riemann sum with two rectangles to estimate the integral.

8. Suppose f is a function of x and y, and that you know

$$\frac{\partial f}{\partial x}(-1,2) = 2\,;$$
$$\frac{\partial f}{\partial y}(-1,2) = 3\,.$$

(a) Sketch a piece of the level set of f that passes through the point $(-1,2)$.



(b) If the x-axis represents **rating** on a scale of -5 to 5 and the y-axis represents **efficiency** on a scale of 0 to 10, what is the marginal rate of substitution of efficiency for rating when staying on the level set of f through the point $(-1,2)$?

(c) Suppose $x(t) = t^2 - 5$ and $y(t) = \frac{4}{t}$. What is $\frac{df}{dt}(2)$?

9. In (a)–(c) find constants c and p such that $f(x) \sim cx^p$ as $x \to \infty$.

(a) $f(x) = x \cdot (3x^2 + 2x\ln(x) + 1)$

(b) $f(x) = \dfrac{\sqrt{e^{2x}+1}}{xe^x}$.

(c) EXTRA CREDIT: $f(x) = (x+1)^{1/3} - x$.

(d) Does $\displaystyle\int_1^\infty \frac{\sqrt{e^{2x}+1}}{xe^x}\,dx$ converge or diverge? Please give a reason.

10. Microsoft has 1 gigabuck to finance its new communication technology. It plans to spend x gigabucks for signal power, and to develop the product for y decades before charging for it, which will cost y gigabucks. The value of the product is estimated to be proportional to the square of the signal strength (signal strength is proportional to x), to e^{ky} (due to exponential growth of the market share during the free period) and to $1 - x - y$ (cash remaining provides liquidity of the communication division).

    (a) Write down a formula for the value V of this project. Explain all constants and variables other than x and y and give units where appropriate.

    (b) The choice of x and y that produces the greatest project value the function over what region R?

(c) Solve the optimization problem when $k = 4$. Three quarters of the credit will be given if you set up the problem correctly.

(d) Solve the optimization problem when $k = 2$.

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$
\begin{aligned}
e &\approx 2.7 \\
\ln(2) &\approx 0.7 \\
\ln(10) &\approx 2.3 \\
\log_{10}(2) &\approx 0.3 \\
\log_{10}(3) &\approx 0.48
\end{aligned}
$$

Some other useful quantities to with 1%:

$$
\begin{aligned}
\pi &\approx \frac{22}{7} \\
\sqrt{10} &\approx \pi \\
\sqrt{2} &\approx 1.4 \\
\sqrt{1/2} &\approx 0.7
\end{aligned}
$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

$\int \frac{e^x}{x^n}\,dx$ cannot be simplified

$\int e^{x^a}\,dx$ cannot be simplified unless $a = 1/n$ for some integer n

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x > a > 0)$