Signature | Printed Name

Preliminary Examination, Part I

Thursday, May 2, 2019 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

Score	
1	
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Grader	

1. Let $\mathcal{P}_n$ the space of polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree at most n with real coefficients.

   a) Give a basis for $\mathcal{P}_n$.

   b) If $x_0,\, x_1, \ldots, x_n \in \mathbb{R}$ are distinct points, define the linear map $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by

   $$Lp = (p(x_0),\, p(x_1), \ldots, p(x_n)).$$

   Find the kernel (=nullspace) of L.

   c) Use part b) to show that for any points $y_0,\, y_1, \ldots, y_n \in \mathbb{R}$ there is a unique $p \in \mathcal{P}_n$ with the property that $p(x_j) = y_j, \quad j = 0, 1, \ldots, n$. [NOTE: You are not being asked to find a formula for p.]

2. Find all positive integers c such that there exists a solution in integers to the equation $33x+24y=c$. For the smallest such c, find all integral solutions (x, y) to that equation. Justify your assertions.

3. Let $g(x)$ be continuous for $x \in \mathbb{R}$ and periodic with period 1, so $g(x+1) = g(x)$ for all real x. Let $\hat{g} = \int_0^1 g(x)\,dx$.

   Show that $\displaystyle\lim_{\lambda\to\infty} \int_0^1 g(\lambda x)\,dx = \hat{g}$.

   [SUGGESTION: First consider $\int_0^1 g(\lambda x)\,dx$ where λ is an integer.]

4. a) Let $q(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ where $a_{n-1}, \ldots, a_0$ are complex numbers. Find a positive real number c (depending on the a_j's) such that $|q(z)| \leq c|z|^{n-1}$ for all $|z| > 1$.

b) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Find a positive real R (depending on the coefficients) such that all of the (possibly complex) roots of p are in the disk $|z| \leq R$.

[HINT: You need only find R for the roots with $|z| > 1$. Apply part a)].

5. a) Compute $\iint_{\mathbb{R}^2} \frac{1}{[1+x^2+y^2]^2}\, dxdy.$

b) Compute $\iint_{\mathbb{R}^2} \frac{1}{[1+(2x-y)^2+(x+y)^2]^2}\, dxdy.$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function.

a) If $\operatorname{grad} f = 0$ in an open disk $D \in \mathbb{R}^2$, show that $f =$constant in D.

b) Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $\operatorname{grad} f = 0$ in Ω, show that $f =$constant in Ω.

[EXTRA PAGE FOR WORK]

Signature	Printed Name

Preliminary Examination, Part II

Thursday, May 2, 2019 1:30-4:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Please write your name on both the exam and any extra sheets you may submit.

All problems have equal weight of 10 points.

Score	
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Grader	

7. Compute $K := \oint_C (2xy + y)dx + 2x^2 dy$, where C is the circle $x^2 + y^2 = 1$ traversed counterclockwise.

8. Let G be any group and let $Z(G)$ be its center. If $G/Z(G)$ is cyclic, prove that G is abelian.

9. Let $f(x)$ be a real-valued function with two continuous derivatives for all real x and periodic with period 2π. Let

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

a) Show there is a constant M (depending on f) so that $|c_k| \le \frac{M}{k^2}$ for all k. [HINT: Integrate by parts.]

b) Show that the series $\sum_{-\infty}^{\infty} c_k e^{ikx}$ converges absolutely and uniformly.

10. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c & 0 \end{pmatrix}$, where c is a real number.

a) For which $c \in \mathbb{R}$ can you diagonalize A over the field of real numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize A].

b) For which $c \in \mathbb{R}$ can you diagonalize A over the field of complex numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize A].

11. a) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with $f(t) \neq 0$ for all t near t_0. Use the definition of the derivative as the limit of a difference quotient to show that $1/f(t)$ is differentiable at t_0.

b) Let $A(t)$ be a square matrix whose elements are infinitely differentiable functions of $t \in \mathbb{R}$. Assume that $A(t)$ is invertible for all t near t_0. Use the definition of the derivative as the limit of a difference quotient to show that $A^{-1}(t)$ is differentiable at t_0.

12. Let A be a real anti-symmetric matrix (so $A^T = -A$) and let $\langle x,\, y\rangle$ be the usual inner product in $\mathbb{R}^n$ (often written $x \cdot y$).

a) Show that $\langle x,\, Ax\rangle = 0$ for all vectors x.

b) If the vector $x(t)$ is a solution of $\dfrac{dx}{dt} = Ax$, show that $\|x(t)\|^2 = \text{constant}$.
[HINT: Use part a).]

[EXTRA PAGE FOR WORK]