This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. Let \( P_n \) the space of polynomials \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) of degree at most \( n \) with real coefficients.
   a) Give a basis for \( P_n \).
      Solution: Basis: 1, \( x \), \( x^2 \), \ldots, \( x^n \) so the dimension is \( n + 1 \).
   b) If \( x_0, x_1, \ldots, x_n \in \mathbb{R} \) are distinct points, define the linear map \( L : P_n \to \mathbb{R}^{n+1} \) by
      \[
      Lp = (p(x_0), p(x_1), \ldots, p(x_n)).
      \]
      Find the kernel (=nullspace) of \( L \).
      Solution: If \( Lp = 0 \) then \( p \) is zero at the \( n + 1 \) points \( x_j, j = 0, 1, \ldots, n \). But a polynomial of degree \( n \) has only \( n \) zeroes – unless it is the zero polynomial.
   c) Use part b) to show that for any points \( y_0, y_1, \ldots, y_n \in \mathbb{R} \) there is a unique \( p \in P_n \) with the property that \( p(x_j) = y_j, \ j = 0, 1, \ldots, n \). [Note: You are not being asked to find a formula for \( p \).]
      Solution: Note \( L : P_n \to \mathbb{R}^{n+1} \) and \( \dim P_n = \dim \mathbb{R}^{n+1} = n+1 \). Since \( \ker(L) = 0 \), \( L \) is invertible.
      Alternate: Newton’s approach gives a natural inductive proof:
      For \( n = 0 \) this is obvious: \( p(x) = y_0 \). Say for any \( y_0, \ldots, y_n \), there is a (unique) \( p \in P_n \) with \( p(x_j) = y_j, \ j = 0, \ldots, n \). Then given \( y_0, \ldots, y_n, y_{n+1} \in \mathbb{R}^{n+2} \) seek \( \hat{p} \in P_{n+1} \) in the form
      \[
      \hat{p}(x) = p(x) + C(x - x_0)(x - x_1)\cdots(x - x_n).
      \]
      Clearly \( \hat{p}(x_j) = y_j \) for \( j = 0, \ldots, n \). The constant \( C \) can now be chosen to satisfy the additional condition \( \hat{p}(x_{n+1}) = y_{n+1} \).
      One could also use the Lagrange basis of \( P_n \) for an explicit construction.
2. Find all positive integers c such that there exists a solution in integers to the equation \(33x + 24y = c\). For the smallest such c, find all integral solutions \((x, y)\) to that equation. Justify your assertions.

**SOLUTION:** Since \(33x + 24y = 3(11x + 8y)\), any such c must be a multiple of 3. Because 11 and 8 are relatively prime, the equation \(11x + 8y = 1\) has a solution, \(x = 3, y = -4\) so the smallest such c = 3.

If \(\hat{x}\) and \(\hat{y}\) is another solution of \(11\hat{x} + 8\hat{y} = 1\), then \(11(x - \hat{x}) + 8(y - \hat{y}) = 0\). Let \(u = x - \hat{x}, v = y - \hat{y}\). Then \(11u = -8v\) so \(u = 8k\) and \(v = -11k\) for any integer \(k\). Consequently, all solutions \((x, y)\) of \(33x + 24y = 3\) have the form \(x = 3 + 8k, y = -(4 + 11k)\).

3. Let \(g(x)\) be continuous for \(x \in \mathbb{R}\) and periodic with period 1, so \(g(x + 1) = g(x)\) for all real \(x\). Let \(\hat{g} = \int_0^1 g(x) \, dx\).

Show that \(\lim_{\lambda \to \infty} \int_0^1 g(\lambda x) \, dx = \hat{g}\).

**[SUGGESTION: First consider \(\int_0^1 g(\lambda x) \, dx\) where \(\lambda\) is an integer.]**

**SOLUTION:** Let \(t = \lambda x\). Then
\[
\int_0^1 g(\lambda x) \, dx = \frac{1}{\lambda} \int_0^\lambda g(t) \, dt.
\]

If \(\lambda = n\) is an integer, the result is obvious from the periodicity of \(g\).

Say \(n \leq \lambda < n + 1\). Let \(M = \max_{x \in \mathbb{R}} |g(x)|\). Then
\[
\frac{1}{\lambda} \int_0^\lambda g(t) \, dt = \frac{1}{\lambda} \int_0^n g(t) \, dt + \frac{1}{\lambda} \int_n^\lambda g(t) \, dt = A + B.
\]

But \(A = \frac{n\hat{g}}{\lambda} \to \hat{g}\) as \(\lambda \to \infty\), while \(|B| \leq \frac{M}{\lambda} \to 0\).

4. a) Let \(q(z) = a_{n-1}z^{n-1} + \cdots + a_1z + a_0\) where \(a_{n-1}, \ldots, a_0\) are complex numbers. Find a positive real number \(c\) (depending on the \(a_j\)'s) such that \(|q(z)| \leq c|z|^{n-1}\) for all \(|z| > 1\).

**SOLUTION:** If \(|z| > 1\), then \(|z^j| \leq |z|^{n-1}\) for all \(0 \leq j \leq n - 1\) so
\[
|q(z)| \leq |a_{n-1}| |z|^{n-1} + |a_{n-2}| |z|^{n-1} + \cdots + |a_1| |z|^{n-1} + |a_0| |z|^{n-1} = \left[|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| \right]|z|^{n-1}.
\]

b) Let \(p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0\). Find a positive real \(R\) (depending on the coefficients) such that all of the (possibly complex) roots of \(p\) are in the disk \(|z| \leq R\).

**[HINT: You need only find \(R\) for the roots with \(|z| > 1\). Apply part a)]**.

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Solution: Say $p(z) = 0$ for some $|z| > 1$. Then by part a)

$$|z^n| = |q(z)| \leq \left[|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0| \right] |z|^{n-1}.$$ 

Thus

$$|z| \leq |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|.$$ 

Since this assumed that $|z| \geq 1$, we conclude that all the roots are in the disk $|z| \leq R$ with

$$R = \max(1, |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|).$$

5. a) Compute $\int\int_{\mathbb{R}^2} \frac{1}{1 + x^2 + y^2} dxdy$.

Solution: In polar coordinates this is

$$\int\int_{\mathbb{R}^2} \frac{1}{1 + r^2} r dr d\theta = 2\pi \int_0^\infty \frac{r dr}{(1 + r^2)^2} = 2\pi \frac{1}{2} = \pi$$

(we used the substitution $u = 1 + r^2$).

b) Compute $\int\int_{\mathbb{R}^2} \frac{1}{1 + (2x - y)^2 + (x + y)^2} dxdy$.

Solution: Making the change of variable $u = 2x - y$, $v = x + y$, since $dudv = \det(2 -1 1 1) dxdy = 3 dxdy$, this integral becomes

$$\frac{1}{3} \int\int_{\mathbb{R}^2} \frac{1}{1 + u^2 + v^2} dudv = \frac{\pi}{3}$$

where we used the result of part a).

6. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be an infinitely differentiable function.

a) If $\text{grad} f = 0$ in an open disk $D \in \mathbb{R}^2$, show that $f = \text{constant}$ in $D$.

Solution: Version 1. Let $p$ be the center of $D$ and $q$ another point of $D$. For $0 \leq t \leq 1$ define $\varphi(t) = f(p + t(q - p))$. Then by the chain rule

$$\varphi'(t) = \text{grad} f(p + t(q - p)) \cdot (q - p) = 0$$

for all $0 \leq t \leq 1$. Thus by the mean value theorem $\varphi(1) = \varphi(0)$, that is, $f(q) = f(p)$ for all $q$ in the disk.

Version 2. Let $P = (a, b)$ be the center of the disk and $Q = (x, y)$ any other point of $D$. Since $\text{grad} f = (f_x, f_y) = 0$, we know that $f_x = 0$ and $f_y = 0$. Thus by the mean value theorem $f$ is constant on both horizontal and vertical lines in $D$. Let $M = (x, b)$ and consider the line segments from $P$ to $M$ and $M$ to $Q$. Since $f$ is constant on both of these segments, then $f(Q) = f(M) = f(P)$. 

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b) Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $\text{grad} \, f = 0$ in $\Omega$, show that $f = \text{constant}$ in $\Omega$.

Solution: Pick a point $P \in \Omega$ and let $S = \{Q \in \Omega \mid f(Q) = f(P)\}$. By part a) the set $S$ is open. To show that $S$ is closed, say $Q_j \in S$ converges to some $\hat{Q} \in \Omega$. Because $f$ is continuous, $f(P) = f(Q_j) \to f(\hat{Q})$. Thus $\hat{x} \in S$.

Since $S \subset \Omega$ is open, closed, and not empty, and $\Omega$ is connected, then $S = \Omega$. 


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7. Compute $K := \oint_C (2xy + y)dx + 2x^2dy$, where $C$ is the circle $x^2 + y^2 = 1$ traversed counterclockwise.

**Solution:** Method 1. Use Stokes’ Theorem in a region $D \subset \mathbb{R}^2$ with oriented boundary $C$:

$$\oint_C p \, dx + q \, dy = \iint_D (q_x - p_y) \, dxdy.$$  

to find

$$K = \iint_D [4x - (2x + 1)] \, dxdy = \iint_D (2x - 1) \, dxdy = -\pi$$  

(since $x$ is an odd function, its integral over $D$ is zero).

Method 2. In polar coordinates on $C$: $x = \cos t$, $y = \sin t$, so

$$(2xy + y) \, dx = (2 \cos t \sin t + \sin t)(-\sin t \, dt) \quad \text{and} \quad 2x^2 \, dy = 2 \cos^2 t \cos t \, dt.$$  

Thus

$$K = \int_0^{2\pi} [( -2 \cos t \sin^2 t - \sin^2 t) + 2 \cos^3 t] \, dt = \int_0^{2\pi} -\sin^2 t \, dt = -\pi.$$  

8. Let $G$ be any group and let $Z(G)$ be its center. If $G/Z(G)$ is cyclic, prove that $G$ is abelian.

**Solution:** Since $G/Z(G)$ is cyclic, denote the generator by $xZ(G)$ for some $x \in G$. Then

$$G = \bigcup_{k \in \mathbb{Z}} x^kZ(G).$$  

For $g_1 = x^{k_1}h_1$ and $g_2 = x^{k_2}h_2$ with $h_i \in Z(G)$, we have

$$g_1g_2 = x^{k_1}h_1x^{k_2}h_2 = x^{k_1+k_2}h_1h_2 = g_2g_1.$$  

So $G$ is abelian.
9. Let $f(x)$ be a real-valued function with two continuous derivatives for all real $x$ and periodic with period $2\pi$. Let

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} \, dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

a) Show there is a constant $M$ (depending on $f$) so that $|c_k| \leq \frac{M}{k^2}$ for all $k$. [HINT: Integrate by parts.]

**SOLUTION:** Integrate by parts twice. Because $f$ and its derivatives are periodic, the boundary terms cancel. Thus

$$c_k = -\frac{1}{2\pi k^2} \int_{-\pi}^{\pi} f''(t)e^{-ikt} \, dt.$$ 

Consequently

$$|c_k| \leq \frac{M}{k^2}, \quad \text{where} \quad M = \max_{|t| \leq \pi} |f''(t)|.$$ 

b) Show that the series $\sum_{-\infty}^{\infty} c_k e^{ikx}$ converges absolutely and uniformly.

**SOLUTION:** Since $\sum \frac{1}{k^2}$ converges, this is a consequence of the Weierstrass M test.

10. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c & 0 \end{pmatrix}$, where $c$ is a real number.

a) For which $c \in \mathbb{R}$ can you diagonalize $A$ over the field of real numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize $A$]

**SOLUTION:** det($A - \lambda I$) = $(-\lambda)(\lambda^2 - c)$.

Case 1, $c > 0$: The eigenvalues are $0, \pm \sqrt{c}$ which are real and distinct so there are 3 real distinct real eigenvectors. Thus $A$ can be diagonalized over the real numbers.

Case 2, $c = 0$: $A \neq 0$ is nilpotent so it cannot be diagonalized.

More directly, all of the eigenvalues of $A$ are 0 but ker $A$ only has dimension 1. Thus $A$ cannot be diagonalized.

Case 3, $c < 0$: The roots of the characteristic polynomial are $\lambda = 0$ and the complex roots $\lambda = \pm \sqrt{-c}i$. Because there is only one real eigenvalue, the matrix cannot be diagonalized over the real numbers.

b) For which $c \in \mathbb{R}$ can you diagonalize $A$ over the field of complex numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize $A$]

**SOLUTION:** The cases $c > 0$ and $c = 0$ are the same as in part a).

If $c < 0$ the roots of the characteristic polynomial are still $\lambda = 0$ and $\lambda = \pm \sqrt{-c}i$. These are distinct so now there are three distinct eigenvectors. Thus $A$ can be diagonalized over the complex numbers.
11. a) Let \( f : \mathbb{R} \to \mathbb{R} \) be an infinitely differentiable function with \( f(t) \neq 0 \) for all \( t \) near \( t_0 \). Use the definition of the derivative as the limit of a difference quotient to show that \( \frac{1}{f(t)} \) is differentiable at \( t_0 \).

**Solution:** Write \( \frac{1}{f(t)} \) as \( f^{-1}(t) \). Then

\[
\frac{f^{-1}(t_0 + h) - f^{-1}(t_0)}{h} = f^{-1}(t_0 + h) \left( \frac{f(t_0) - f(t_0 + h)}{h} \right) f^{-1}(t_0),
\]

so

\[
\lim_{h \to 0} \frac{f^{-1}(t_0 + h) - f^{-1}(t_0)}{h} = f^{-1}(t_0) \left( -f'(t_0) \right) f^{-1}(t_0) = -\frac{f'(t_0)}{f^2(t_0)}.
\]

b) Let \( A(t) \) be a square matrix whose elements are infinitely differentiable functions of \( t \in \mathbb{R} \). Assume that \( A(t) \) is invertible for all \( t \) near \( t_0 \). Use the definition of the derivative as the limit of a difference quotient to show that \( A^{-1}(t) \) is differentiable at \( t_0 \).

**Solution:** We follow part a) closely:

\[
\frac{A^{-1}(t_0 + h) - A^{-1}(t_0)}{h} = A^{-1}(t_0 + h) \left( \frac{A(t_0) - A(t_0 + h)}{h} \right) A^{-1}(t_0),
\]

so

\[
\lim_{h \to 0} \frac{A^{-1}(t_0 + h) - A^{-1}(t_0)}{h} = A^{-1}(t_0) \left( -A'(t_0) \right) A^{-1}(t_0) = -A^{-1}(t_0)A'(t_0)A^{-1}(t_0).
\]

12. Let \( A \) be a real anti-symmetric matrix (so \( A^T = -A \)) and let \( \langle x, y \rangle \) be the usual inner product in \( \mathbb{R}^n \) (often written \( x \cdot y \)).

a) Show that \( \langle x, Ax \rangle = 0 \) for all vectors \( x \).

**Solution:** \( \langle x, Ax \rangle = \langle A^T x, x \rangle = -\langle Ax, x \rangle = -\langle x, Ax \rangle \).

b) If the vector \( x(t) \) is a solution of \( \frac{dx}{dt} = Ax \), show that \( \|x(t)\|^2 = \text{constant} \).

[HINT: Use part a).]

**Solution:** By part a),

\[
\frac{d\|x(t)\|^2}{dt} = d\langle x, x \rangle = 2\langle x, x' \rangle = 2\langle x, Ax \rangle = 0.
\]