Fall 2018 Preliminary Exam – Problems and Solutions

PRELIMINARY EXAMINATION, PART I

Thursday, May 2, 2019 9:30-12:00

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

All problems have equal weight of 10 points.

1. Let $\mathcal{P}_n$ the space of polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ of degree at most n with real coefficients.

   a) Give a basis for $\mathcal{P}_n$.

   SOLUTION: Basis: $1,\ x,\ x^2, \ldots,\ x^n$ so the dimension is $n+1$.

   b) If $x_0,\ x_1, \ldots, x_n \in \mathbb{R}$ are distinct points, define the linear map $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ by

   $$Lp = (p(x_0),\ p(x_1), \ldots, p(x_n)).$$

   Find the kernel (=nullspace) of L.

   SOLUTION: If $Lp = 0$ then p is zero at the $n+1$ points x_j, $j = 0, 1, \ldots, n$. But a polynomial of degree n has only n zeroes – unless it is the zero polynomial.

   c) Use part b) to show that for any points $y_0,\ y_1, \ldots, y_n \in \mathbb{R}$ there is a unique $p \in \mathcal{P}_n$ with the property that $p(x_j) = y_j, \quad j = 0, 1, \ldots, n$. [NOTE: You are not being asked to find a formula for p.]

   SOLUTION: Note $L : \mathcal{P}_n \to \mathbb{R}^{n+1}$ and $\dim \mathcal{P}_n = \dim \mathbb{R}^{n+1} = n+1$. Since $\ker(L) = 0$, L is invertible.

   ALTERNATE: Newton's approach gives a natural inductive proof:

   For $n = 0$ this is obvious: $p(x) = y_0$. Say for any $y_0, \ldots, y_n$, there is a (unique)) $p \in \mathcal{P}_n$ with $p(x_j) = y_j,\ j = 0, \ldots, n$. Then given $y_0, \ldots, y_n,\ y_{n+1} \in \mathbb{R}^{n+2}$ seek $\hat{p} \in \mathcal{P}_{n+1}$ in the form

   $$\hat{p}(x) = p(x) + C(x - x_0)(x - x_1) \cdots (x - x_n).$$

   Clearly $\hat{p}(x_j) = y_j$ for $j = 0, \ldots, n$. The constant C can now be chosen to satisfy the additional condition $\hat{p}(x_{n+1}) = y_{n+1}$.

   One could also use the Lagrange basis of $\mathcal{P}_n$ for an explicit construction.

2. Find all positive integers c such that there exists a solution in integers to the equation $33x+24y = c$. For the smallest such c, find all integral solutions (x, y) to that equation. Justify your assertions.

SOLUTION: Since $33x+24y = 3(11x+8y)$, any such c must be a multiple of 3. Because 11 and 8 are relatively prime, the equation $11x + 8y = 1$ has a solution, $x = 3$, $y = -4$ so the smallest such $c = 3$.

If $\hat{x}$ and $\hat{y}$ is another solution of $11\hat{x}+8\hat{y} = 1$, then $11(x-\hat{x})+8(y-\hat{y}) = 0$. Let $u = x-\hat{x}$, $v = y - \hat{y}$. Then $11u = -8v$ so $u = 8k$ and $v = -11k$ for any integer k. Consequently, all solutions (x, y) of $33x + 24y = 3$ have the form $x = 3 + 8k$, $y = -(4 + 11k)$.

3. Let $g(x)$ be continuous for $x \in \mathbb{R}$ and periodic with period 1, so $g(x + 1) = g(x)$ for all real x. Let $\hat{g} = \int_0^1 g(x)\,dx$.

Show that $\displaystyle\lim_{\lambda\to\infty} \int_0^1 g(\lambda x)\,dx = \hat{g}$.

[SUGGESTION: First consider $\int_0^1 g(\lambda x)\,dx$ where λ is an integer.]

SOLUTION: Let $t = \lambda x$. Then

$$\int_0^1 g(\lambda x)\,dx = \frac{1}{\lambda}\int_0^\lambda g(t)\,dt.$$

If $\lambda = n$ is an integer, the result is obvious from the periodicity of g.

Say $n \le \lambda < n + 1$. Let $M = \max_{x\in\mathbb{R}}|g(x)|$. Then

$$\frac{1}{\lambda}\int_0^\lambda g(t)\,dt = \frac{1}{\lambda}\int_0^n g(t)\,dt + \frac{1}{\lambda}\int_n^\lambda g(t)\,dt = A + B.$$

But $A = \frac{n\hat{g}}{\lambda} \to \hat{g}$ as $\lambda \to \infty$, while $|B| \le \frac{M}{\lambda} \to 0$.

4. a) Let $q(z) = a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ where $a_{n-1}, \ldots, a_0$ are complex numbers. Find a positive real number c (depending on the a_j's) such that $|q(z)| \le c|z|^{n-1}$ for all $|z| > 1$.

SOLUTION: If $|z| > 1$, then $|z^j| \le |z|^{n-1}$ for all $0 \le j \le n - 1$ so

$$\begin{aligned}|q(z)| \le& |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-1} + \cdots + |a_1||z|^{n-1} + |a_0||z|^{n-1}\\ =&\Big[|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\Big]\,|z|^{n-1}\end{aligned}.$$

b) Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$. Find a positive real R (depending on the coefficients) such that all of the (possibly complex) roots of p are in the disk $|z| \le R$.

[HINT: You need only find R for the roots with $|z| > 1$. Apply part a)].

SOLUTION: Say $p(z) = 0$ for some $|z| > 1$. Then by part a)

$$\begin{aligned} |z^n| =& |q(z)| \\ \leq& \Big[|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|\Big] \, |z|^{n-1}. \end{aligned}$$

Thus

$$|z| \leq |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|.$$

Since this assumed that $|z| \geq 1$, we conclude that all the roots are in the disk $|z| \leq R$ with

$$R = \max(1, \ |a_{n-1}| + |a_{n-2}| + \cdots + |a_0|).$$

5. a) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{1}{[1 + x^2 + y^2]^2} \, dxdy.$

SOLUTION: In polar coordinates this is

$$\iint_{\mathbb{R}^2} \frac{1}{(1 + r^2)^2} \, r dr d\theta = 2\pi \int_0^\infty \frac{r dr}{(1 + r^2)^2} = 2\pi \frac{1}{2} = \pi$$

(we used the substitution $u = 1 + r^2$).

b) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{1}{[1 + (2x - y)^2 + (x + y)^2]^2} \, dxdy.$

SOLUTION: Making the change of variable $u = 2x - y$, $v = x + y$, since $dudv = \det\left(\begin{smallmatrix} 2 & -1 \\ 1 & 1 \end{smallmatrix}\right) dxdy = 3\, dxdy$, this integral becomes

$$\frac{1}{3} \iint_{\mathbb{R}^2} \frac{1}{(1 + u^2 + v^2)^2} \, dudv = \frac{\pi}{3}$$

where we used the result of part a).

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function.

a) If $\operatorname{grad} f = 0$ in an open disk $D \in \mathbb{R}^2$, show that $f =$constant in D.

SOLUTION: Version 1. Let p be the center of D and q another point of D. For $0 \leq t \leq 1$ define $\varphi(t) = f(p + t(q - p))$. Then by the chain rule

$$\varphi'(t) = \operatorname{grad} f(p + t(q - p)) \cdot (q - p) = 0$$

for all $0 \leq t \leq 1$. Thus by the mean value theorem $\varphi(1) = \varphi(0)$, that is, $f(q) = f(p)$ for all q in the disk.

Version 2. Let $P = (a, b)$ be the center of the disk and $Q = (x, y)$ any other point of D. Since $\operatorname{grad} f = (f_x, \, f_y) = 0$, we know that $f_x = 0$ and $f_y = 0$. Thus by the mean value theorem f is constant on both horizontal and vertical lines in D. Let $M = (x, b)$ and consider the line segments from P to M and M to Q. Since f is constant on both of these segments, then $f(Q) = f(M) = f(P)$.

b) Let $\Omega \subset \mathbb{R}^2$ be a connected open set. If $\operatorname{grad} f = 0$ in Ω, show that $f =$constant in Ω.

SOLUTION: Pick a point $P \in \Omega$ and let $S = \{Q \in \Omega \,|\, f(Q) = f(P)\}$. By part a) the set S is open. To show that S is closed, say $Q_j \in S$ converges to some $\hat{Q} \in \Omega$. Because f is continuous, $f(P) = f(Q_j) \to f(\hat{Q})$. Thus $\hat{x} \in S$.

Since $S \subset \Omega$ is open, closed, and not empty, and Ω is connected, then $S = \Omega$.

Thursday, May 2, 2019 1:30-4:00

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7. Compute $K := \oint_C (2xy + y)dx + 2x^2dy$, where C is the circle $x^2 + y^2 = 1$ traversed counterclockwise.

   Solution: Method 1. Use Stokes' Theorem in a region $D \subset \mathbb{R}^2$ with oriented boundary C:
   $$\oint_C pdx + qdy = \iint_D (q_x - p_y)\,dxdy.$$
   to find
   $$K = \iint_D [4x - (2x+1)]\,dxdy = \iint_D (2x-1)\,dxdy = -\pi$$
   (since x is an odd function, its integral over D is zero).

   Method 2. In polar coordinates on C: $x = \cos t$, $y = \sin t$, so
   $$(2xy + y)\,dx = (2\cos t \sin t + \sin t)(-\sin t\,dt) \quad \text{and} \quad 2x^2\,dy = 2\cos^2 t \cos t\,dt.$$
   Thus
   $$K = \int_0^{2\pi} [(-2\cos t \sin^2 t - \sin^2 t) + 2\cos^3 t]\,dt = \int_0^{2\pi} -\sin^2 t\,dt = -\pi.$$

8. Let G be any group and let $Z(G)$ be its center. If $G/Z(G)$ is cyclic, prove that G is abelian.

   Solution: Since $G/Z(G)$ is cyclic, denote the generator by $xZ(G)$ for some $x \in G$. Then
   $$G = \bigcup_{k\in\mathbb{Z}} x^k Z(G).$$
   For $g_1 = x^{k_1}h_1$ and $g_2 = x^{k_2}h_2$ with $h_i \in Z(G)$, we have
   $$g_1g_2 = x^{k_1}h_1x^{k_2}h_2 = x^{k_1+k_2}h_1h_2 = g_2g_1.$$
   So G is abelian.

9. Let $f(x)$ be a real-valued function with two continuous derivatives for all real x and periodic with period 2π. Let

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

a) Show there is a constant M (depending on f) so that $|c_k| \le \frac{M}{k^2}$ for all k. [HINT: Integrate by parts.]

SOLUTION: Integrate by parts twice. Because f and its derivatives are periodic, the boundary terms cancel. Thus

$$c_k = \frac{-1}{2\pi k^2}\int_{-\pi}^{\pi} f''(t)e^{-ikt}\,dt.$$

Consequently

$$|c_k| \le \frac{M}{k^2}, \qquad \text{where} \qquad M = \max_{|t|\le\pi}|f''(t)|.$$

b) Show that the series $\sum_{-\infty}^{\infty} c_k e^{ikx}$ converges absolutely and uniformly.

SOLUTION: Since $\sum \frac{1}{k^2}$ converges, this is a consequence of the Weierstrass M test.

10. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & c & 0 \end{pmatrix}$, where c is a real number.

a) For which $c \in \mathbb{R}$ can you diagonalize A over the field of real numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize A].

SOLUTION: $\det(A - \lambda I) = (-\lambda)(\lambda^2 - c)$.

Case 1, $c > 0$: The eigenvalues are 0, $\pm\sqrt{c}$ which are real and distinct so there are 3 real distinct real eigenvectors. Thus A can be diagonalized over the real numbers.

Case 2, $c = 0$: $A \ne 0$ is nilpotent so it cannot be diagonalized.
More directly, all of the eigenvalues of A are 0 but $\ker A$ only has dimension 1. Thus A cannot be diagonalized.

Case 3, $c < 0$: The roots of the characteristic polynomial are $\lambda = 0$ and the complex roots $\lambda = \pm\sqrt{-c}\,i$. Because there is only one real eigenvalue, the matrix cannot be diagonalized over the real numbers.

b) For which $c \in \mathbb{R}$ can you diagonalize A over the field of complex numbers? Explain your reasoning. [Note: all you are being asked is IF you can diagonalize A].

SOLUTION: The cases $c > 0$ and $c = 0$ are the same as in part a).
If $c < 0$ the roots of the charachteristic polynomial are still $\lambda = 0$ and $\lambda = \pm\sqrt{-c}\,i$. These are distinct so now there are three distinct eigenvectors. Thus A can be diagonalized over the complex numbers.

11. a) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with $f(t) \neq 0$ for all t near t_0. Use the definition of the derivative as the limit of a difference quotient to show that $1/f(t)$ is differentiable at t_0.

   SOLUTION: Write $1/f(t)$ as $f^{-1}(t)$. Then

$$\frac{f^{-1}(t_0+h) - f^{-1}(t_0)}{h} = f^{-1}(t_0+h)\left(\frac{f(t_0) - f(t_0+h)}{h}\right) f^{-1}(t_0),$$

   so

$$\lim_{h\to 0} \frac{f^{-1}(t_0+h) - f^{-1}(t_0)}{h} = f^{-1}(t_0)\Big(-f'(t_0)\Big) f^{-1}(t_0) = -\frac{f'(t_0)}{f^2(t_0)}.$$

   b) Let $A(t)$ be a square matrix whose elements are infinitely differentiable functions of $t \in \mathbb{R}$. Assume that $A(t)$ is invertible for all t near t_0. Use the definition of the derivative as the limit of a difference quotient to show that $A^{-1}(t)$ is differentiable at t_0.

   SOLUTION: We follow part a) *closely*:

$$\frac{A^{-1}(t_0+h) - A^{-1}(t_0)}{h} = A^{-1}(t_0+h)\left(\frac{A(t_0) - A(t_0+h)}{h}\right) A^{-1}(t_0),$$

   so

$$\lim_{h\to 0} \frac{A^{-1}(t_0+h) - A^{-1}(t_0)}{h} = A^{-1}(t_0)\Big(-A'(t_0)\Big) A^{-1}(t_0) = -A^{-1}(t_0)A'(t_0)A^{-1}(t_0).$$

12. Let A be a real anti-symmetric matrix (so $A^T = -A$) and let $\langle x,\, y\rangle$ be the usual inner product in $\mathbb{R}^n$ (often written $x \cdot y$).

   a) Show that $\langle x,\, Ax\rangle = 0$ for all vectors x.

   SOLUTION: $\langle x,\, Ax\rangle = \langle A^T x,\, x\rangle = -\langle Ax,\, x\rangle = -\langle x,\, Ax\rangle$.

   b) If the vector $x(t)$ is a solution of $\dfrac{dx}{dt} = Ax$, show that $\|x(t)\|^2 = \text{constant}$.

   [HINT: Use part a).]

   SOLUTION: By part a),

$$\frac{d\|x(t)\|^2}{dt} = \frac{d\langle x,\, x\rangle}{dt} = 2\langle x,\, x'\rangle = 2\langle x,\, Ax\rangle = 0.$$