Sample Preliminary Examination

1. a) Find all solutions in integers to the equation $129x+291y=1$.

   b) Do the same for the equation $129x+291y=3$.

   Justify your assertions.

2. Show $f(x)=x^2$ is not uniformly continuous as a function on the whole real line (i.e. show for some $\epsilon>0$ there is no $\delta>0$ so that $|f(x)-f(y)|<\epsilon$ whenever $|x-y|<\delta$).

3. For each of the following, either give an example or explain why none exists.

   a) A non-abelian group of order 20.

   b) Two non-isomorphic abelian groups of order 30.

   c) A finite field whose non-zero elements form a cyclic group of order 17 under multiplication.

   d) A non-trivial automorphism of a finite field.

4. Let f be a real-valued continuous function defined for all $0 \leq x \leq 1$, such that $f(0)=1$, $f(1/2)=2$ and $f(1)=3$. Show that

$$\lim_{n\to\infty}\int_0^1 f(x^n)dx$$

   exists and compute this limit. Justify your assertions.

5. Let V be the real vector space consisting of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most 5 (including the 0 polynomial).

   a) Find a basis for V, and determine the dimension of V.

   b) Define $T\colon V \to \mathbb{R}^6$ by $T(f)=(f(0),f(1),f(2),f(3),f(4),f(5))$. Show T is a linear transformation and find its kernel.

   c) Deduce that for every choice of $a_0,\cdots,a_5 \in \mathbb{R}$ there is a unique polynomial $f(x)\in\mathbb{R}[x]$ of degree at most 5 such that $f(j)=a_j$ for $j=0,1,\cdots,5$.

6. a) Is there a metric space structure on the set $\mathbb{Z}$ such that the open sets are precisely the subsets $S \subset \mathbb{Z}$ such that $\mathbb{Z} - S$ is finite, and also the empty set?

   b) Is there a metric space structure on the set $\mathbb{Z}$ such that every subset is open?

   Justify your assertions.

7. Let $\vec{F}$ be a vector field defined in $\mathbb{R}^3$ minus the origin defined by

$$\vec{F}(\vec{r}) = \frac{\vec{r}}{\| \vec{r} \|^3} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

   for $\vec{r} \neq 0$.

   a) Compute $div\ \vec{F}$.

   b) Let S be the sphere of radius 1 centered at $(x, y, z) = (2, 0, 0)$. Compute

$$\iint_S \vec{F} \cdot \vec{n} \quad dS.$$

8. Let $\{a_n\}$ be a bounded sequence of real numbers. Consider the infinite series

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n}$$

   where x is a real number. Prove that for any $c > 1$ this series converges uniformly on $\{x \in \mathbb{R} \mid x \geq c\}$.

9. Let A be the ring of continuous functions $f \colon \mathbb{R} \to \mathbb{R}$, under (pointwise) addition and multiplication.

   a) Determine whether A is a integral domain.

   b) Let $I \subset A$ be the subset consisting of functions f such that $f(0) = 0$. Is I an ideal? Is it a maximal ideal? What is A/I?

10. Suppose $\{a_n : n = 1, 2, \cdots\}$ is a sequence of real numbers so that

$$\sum_{n=1}^{\infty} |a_n| = 1.$$

Let $f(x)$ be given by the cos series

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).$$

Prove that the series for f converges and that f is continuous.

11. Let

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Find the minimal and characteristic polynomials of M.

b) Is M similar to a diagonal matrix D over $\mathbb{R}$? If so, find such a D.

c) Repeat part (b) with $\mathbb{R}$ replaced by $\mathbb{C}$ and also by the field $\mathbb{Z}/5\mathbb{Z}$.

12. Let V be the vector space of C^{∞} real-valued functions on $\mathbb{R}$. Consider the following maps $T_i : V \to V$.

$$T_1(f) = f'' - 6f' + 9f$$
$$T_2(f) = f' - xf$$
$$T_3(f) = ff'$$

a) Which of the maps T_i are linear transformations?

b) For each one that is, find a basis for the kernel.