Premiliminary Examination, Sample Exam

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Problem 1. *(a) Find all solutions in integers to the equaiton* $129x + 291y = 1$
(b) Do the same for the equation $129x + 291y = 3$
Justify your assertions.

Solution. (a) There are no solutions. Indeed, since $3 \mid 129$ and $3 \mid 291$ we have that $3 \mid 129x + 291y$, however $3 \nmid 1$.

(b) Suppose we have a solution, i.e. a pair (x_0, y_0) that satisfies $43x_0 + 97y_0 = 1$. Then any other solutions is of the form $x = x_0 + 97m$, $y = y_0 - 43m$ for $m \in \mathbb{Z}$. Indeed, suppose (x_1, y_1) is another solution then subtracting the two equations we obtain

$$43(x_1 - x_0) + 97(y_1 - y_0) = 0 \tag{1}$$

Since $\gcd(43, 97) = 1$, taking equation (1) modulo 97 and 43 we find that $x_1 - x_0 = 97m_1$ and $y_1 - y_0 = 43m_2$. Plugging these two expressions into (1), we get $43 \cdot 97m_1 + 43 \cdot 97m_2 = 0$, hence $m_2 = -m_1$.

Finally we need to determine a special solution. By a variation of the Euclid's algorithm we find that $(-9, 4)$ is a special solution. Therefore, the general solution is given by $s_m = (-9 + 97m, 4 - 43m)$. ■

Problem 2. *Show that* $f(x) = x^2$ *is not uniformly continuous as a function on the whole real line (i.e. show for some* $\epsilon > 0$ *there is no* $\delta > 0$ *so that* $|f(x) - f(y)| < \epsilon$ *whenever* $|x - y| < \delta$*).*

Solution. Fix $\epsilon > 0$ and $\delta > 0$. It suffices to show that there are x, y such that $|x - y| < \delta$ and $|f(x) - f(y)| > \epsilon$. To that end, let $x = \frac{1}{\delta}\epsilon + \frac{\delta}{2}$ and $y = \frac{1}{\delta}\epsilon$. So,

$$\begin{aligned} |f(x) - f(y)| &= \tfrac{\delta}{2}\left(\tfrac{2}{\delta}\epsilon + \tfrac{\delta}{2}\right) \\ &> \tfrac{\delta}{2} \cdot \tfrac{2}{\delta}\epsilon \\ &= \epsilon \end{aligned}$$

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Problem 3. *For each of the following, either give an example or explain why none exists.*

(a) A non-abelian group of order 20.
(b) Two non-isomorphic abelian groups of order 30.
(c) A finite field whose non-zero elements form a cyclic group of order 17 under multiplication.
(d) A non-trivial automorphism of a finite field.

Solution. (a) There are non-abelian groups of order 20; one example is the dihedral group of 20 elements. This is the group of symmetries of a regular 10-agon, and, it is usually, depending on the context, denoted by D_{10} or D_{20}.

(b) This is impossible. Since from the fundamental theorem of finitely generated abelian groups, a group with 30 elements is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$.

(c) There is no such field. All finite fields have cardinality p^n where p is a prime. The multiplicative group of a field is cyclic, and has cardinality $p^n - 1$. Therefore, such a field must satisfy $p^n - 1 = 17$, or $p^n = 18$, which is impossible.

(d) There are groups with non trivial automorphisms. Take a field F of characteristic p, with p^n number of elements, where $n > 1$. Take $\phi : F \to F$ given by $a \mapsto a^p$. We show that ϕ is an automorphism. First, using the binomial expansion we see that $\phi(a+b) = \phi(a) + \phi(b)$, and of course $\phi(ab) = \phi(a)\phi(b)$, hence ϕ is a homomorphism. As $\ker \phi = \{0\}$, and F is finite, ϕ is an isomorphism. We claim that ϕ is not trivial. Indeed the multiplicative group of a field is cyclic, with order $p^n - 1$. Hence, there is an $a \in F$ such that $a^p \neq a$. ■

Problem 4. *Lef f be a real-valued continuous function defined for all $0 \le x \le 1$, such that $f(0) = 1, f(1/2) = 2$ ad $f(1) = 3$. Show that*

$$\lim_{n\to\infty} \int_0^1 f(x^n)\,\mathrm{d}x$$

exists and compute this limit. Justify your assertions.

Solution. The limit is equal to $\int_0^1 f(0)\,\mathrm{d}x = 1$. Let $\epsilon > 0$. We find $\delta > 0$ such that $|f(x) - f(0)| < \frac{\epsilon}{2}$ for all $x \in [0, \delta)$. Now, pick $\delta_1 > 0$ and N such that $\int_{1-\delta_1}^1 \max_{x\in[0,1]} |f(x) - 1|\,\mathrm{d}x < \frac{\epsilon}{2}$ and $(1-\delta_1)^n < \delta$ for all $n \ge N$.

Then for $n \ge N$ we get the following

$$\begin{aligned}
\int_0^1 |f(x^n) - 1|\,\mathrm{d}x &= \int_0^{1-\delta_1} |f(x^n) - 1|\,\mathrm{d}x + \int_{1-\delta_1}^1 |f(x^n) - 1|\,\mathrm{d}x \\
&\le \int_0^{1-\delta_1} \frac{\epsilon}{2}\,\mathrm{d}x + \int_{1-\delta_1}^1 \max_{x\in[0,1]} |f(x) - 1| \\
&= (1-\delta_1)\frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

And since $\left|\int_0^1 f(x^n)\,\mathrm{d}x - 1\right| = \left|\int_0^1 (f(x^n) - 1)\,\mathrm{d}x\right| \le \int_0^1 |f(x^n) - 1|\,\mathrm{d}x$ we conclude.

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Problem 5. *Let V be the real vector space consisting of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most 5 (including the 0 polynomial).*

(a) Find a basis for V , and determine the dimension of V.

(b) Define $T : V \to \mathbb{R}^6$ by $T(f) = (f(0), f(1), f(2), f(3), f(4), f(5))$. Show that T is a linear transformation and find its kernel.

(c) Deduce that for every choice of $a_0, a_1, \ldots, a_5 \in \mathbb{R}$ there is a unique polynomial $f(x) \in \mathbb{R}[x]$ of degree at most 5 such that $f(j) = a_j$ for $j = 0, 1, \ldots, 5$.

Solution. (a) We have the following description for $V = \{a_6x^n + a_5x^5 + \cdots + a_1x + a_0 | a_i \in \mathbb{R}, \text{ for } 1 \leq i \leq 6\}$, the canonical basis is $e_i = x^i$. From the definition of V we gave, $V =< e_1, e_2, \ldots, e_6 >$. To see why e_i are linearly independent take λ_i for $1 \leq i \leq 6$ such that

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_6 e_6 = 0. \tag{2}$$

The polynomial $p(x)$ defined at the LSH of equation (2), must be the zero polynomial since only the zero polynomial has infinite many roots. The coefficients of $p(x)$ are exactly the scalars λ_i, therefore $\lambda_i = 0$ for all i, which in turns establishes that e_i are linearly independent.

(b) The operator T is linear, indeed,

$$\begin{aligned} T(\lambda f + \mu g) &= ((\lambda f + \mu g)(0), (\lambda f + \mu g)(1), \ldots, (\lambda f + \mu g)(5)) \\ &= (\lambda f(0) + \mu g(0), \lambda f(1) + \mu g(1), \ldots, \lambda f(5) + \mu g(5)) \\ &= \lambda(f(0), f(1), \ldots, f(5)) + \mu(g(0), g(1), \ldots, g(5)) \\ &= \lambda T(f) + \mu T(g) \end{aligned}$$

The kernel of T is trivial. Indeed, suppose $T(f) = 0$ then $f = 0$ as any non-constant polynomial of degree at most 5 has at most 5 roots.

(c) Suppose f, g are two polynomials in V such that $f(j) = g(j) = a_j$ for all $j = 1, 2, \cdots, 5$. We can express the previous statement via the operator T as $T(f) = T(g)$ which in turn implies $T(f - g) = 0$, therefore $f = g$. ■

Problem 6. *(a) Is there a metric space strucuture on the set $\mathbb{Z}$ such that the open sets are precisely the subsets $S \subset \mathbb{Z}$ such that $\mathbb{Z} - S$ is finite, and also the empty set?*

(b) Is there a metric space structure on the set $\mathbb{Z}$ such that every subset is open?

Justify your assertions.

Solution. (a) No. All metric structures are Hausdorff, however the topology at hand is not. A topology is Hausdorf if for every two points x, y there are open sets V_x, V_y such that $x \in V_x, y \in V_y$ and $V_x \cap V_y = \emptyset$. To see why the topology is not Hausdorff, notice that for V a non-trivial open set there is a $M \in \mathbb{Z}$ such that $\{M, M+1, \cdots, M+n, \cdots\} \subset V$, hence any two open sets (non-empty) V, U intersect non-trivially.

(b) Yes. The discrete metric d, defined by $\mathrm{d}(x, y) = 1$ if $x \neq y$ and zero otherwise, induces a topology such that every subset is open. To see this note that the singletons are open, and recall that union of open sets is open. ■

Problem 7. *Let $\vec{F}$ be a vector field defined in $\mathbb{R}^3$ minus the origin defined by*

$$\vec{F}(\vec{r}) = \frac{\vec{r}}{||\vec{r}||^3} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

for $\vec{r} \neq 0$.

(a) Compute $\operatorname{div} \vec{F}$.

(b) Let S be the sphere of radius 1 centered at $(x, y, z) = (2, 0, 0)$. Compute

$$\oiint_S \vec{F} \cdot \vec{n} \, dS \, .$$

Solution. (a) By definition of the divergence operator we have

$$\begin{aligned}\operatorname{div}\vec{F} &= \frac{\partial(x/||\vec{r}||^3)}{\partial x} + \frac{\partial(y/||\vec{r}||^3)}{\partial y} + \frac{\partial(z/||\vec{r}||^3)}{\partial z} \\ &= 3\left(||\vec{r}||^{-3} - \frac{x^2}{||\vec{r}||^5} - \frac{y^2}{||\vec{r}||^5} - \frac{z^2}{||\vec{r}||^5}\right) \\ &= 0\end{aligned}$$

(b) Since the singular point of the vector field is not inside $\mathrm{Conv}(S)$ the convex hull of S, we can apply the divergence theorem.

$$\begin{aligned}\oint\!\!\!\oint_S \vec{F}\cdot\vec{n}\,\mathrm{d}S &= \iiint_{\mathrm{Conv}(S)} \operatorname{div}\vec{F}\,\mathrm{d}V \\ &= \iiint_{\mathrm{Conv}(S)} 0\,\mathrm{d}V \\ &= 0\end{aligned}$$

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Problem 8. *Let $\{a_n\}$ be a bounded sequence of real numbers. Consider the infinite series*

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n}$$

where x is a real number. Prove that for any $c > 1$ this series converges uniformly on $\{x \in \mathbb{R} | x \geq c\}$.

Solution. Define the power series $p(x) = \sum_{n=1}^{\infty} a_n x^n$. To find R, its radius of convergence, we calculate $\limsup \sqrt[n]{|a_n|}$. The sequence a_n is bounded; so there is $M > 0$ such that $|a_n| < M$, and, since $\sqrt[n]{M} \longrightarrow 1$, we conclude that $\limsup \sqrt[n]{|a_n|} \leq 1$. So, $R = \frac{1}{\limsup \sqrt[n]{|a_n|}} \geq 1$.

For every $\delta > 0$, a power series with radius of convergence R convergences uniformly on $(R-\delta, R+\delta)$. Therefore, for any c such that $0 < \frac{1}{c} < 1$, $p(x)$ converges uniformly on $A = (0, \frac{1}{c}]$. Since $f(1/x) = p(x)$, f converges uniformly on $\frac{1}{x}((0, \frac{1}{c}]) = [c, \infty)$ as desired. ■

Problem 9. *Let A be the ring of continuous functions $f : \mathbb{R} \to \mathbb{R}$, under (pointwise) addition and multiplication.*

(a) Determine whether A is an integral domain.

(b) Let $I \subset A$ be the subset consisting of functions f such that $f(0) = 0$. Is I an ideal? What is A/I?

Solution. (a) The ring A is not an integral domain. Indeed, take $f^+(x) = x \cdot 1_{[0,\infty)}(x)$ and $f^-(x) = x \cdot 1_{(-\infty,0]}(x)$, then $f^+ \cdot f^- = 0$.

(b) Yes, as the I has additive subgroup structure, since $(f-g)(0) = 0$ for all $f, g \in I$; and the multiplication is absorbing, i.e. $(rf)(0) = r(0) \cdot 0 = 0$ for all continuous functions r. The ring A/I is isomorphic to $\mathbb{R}$, to see this define $\phi : A/I \to \mathbb{R}$ where $\phi([f]) = f(0)$. First, the map ϕ is well-defined since $[f] = [g] \iff f(0) = g(0)$. Furthermore, ϕ is a ring homomorphism since $\phi([fg]) = f(0)g(0) = \phi([f])\phi([g])$, and $\phi([f+g]) = f(0) + g(0) = \phi([f]) + \phi([g])$.

We show that ϕ is a bijection. For surjectivity notice that any constant map c is continuous. Now, to prove that ϕ is injective we calculate its kernel: $\phi([f]) = 0 \iff f(0) = 0 \iff f \in I$, therefore $\ker \phi = \{[0]\}$. Hence, indeed, ϕ is a ring isomorphism. ■

Problem 10. *Suppose $\{a_n : n = 1, n = 2, \ldots\}$ is a sequence of real numbers so that*

$$\sum_{n=1}^{\infty} |a_n| = 1.$$

Let $f(x)$ be given by the cos *series*

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).$$

Prove that the series for f converges and that f is continuous.

Solution. Define $S_m = \sum_{n=1}^{m-1} a_n \cos(nx)$. Firstly, notice that f exists since the series $\sum_{n=1}^{\infty} a_n \cos(nx)$ is absolutely convergent. To show that f is continuous, it suffices to show that S_m converges uniformly to f, since uniform convergence preserves continuity. Let $\epsilon > 0$, and choose N such that $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Then, for all $m \geq N$

$$\begin{aligned}
|f(x) - S_m(x)| &= \left| \sum_{n=m}^{\infty} a_n \cos(nx) \right| \\
&\leq \sum_{n=m}^{\infty} |a_n| |\cos(nx)| \\
&= \sum_{n=m}^{\infty} |a_n| \\
&\leq \sum_{n=N}^{\infty} |a_n| < \epsilon
\end{aligned}$$

Taking sup over all x establishes the uniform convergence. ■

Problem 11. *Let*

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Find the minimal and characteristic polynomial of M.
(b) Is M similar to a diagonal matrix D over $\mathbb{R}$? If so, find such a D.
(c) Repeat part (b) with $\mathbb{R}$ replaced by $\mathbb{C}$ and also by the field $\mathbb{Z}/5\mathbb{Z}$

Solution. We calculate the characteristic polynomial $\chi_M(x)$ of M by using the Laplace expansion,

$$\begin{aligned}
\begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} &= -1 \cdot \begin{vmatrix} 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{vmatrix} - \lambda \cdot \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} \\
&= \lambda^4 - 1
\end{aligned}$$

So, $\chi_M(\lambda) = \lambda^4 - 1$. To find the μ_M the minimal polynomial of M we distinguish two cases. First assume that the underlying field is not of characteristic 2. In this case $\gcd(X'_M, X_M) = 1$ which shows that X_M does not have double roots, therefore $\mu_M = \chi_M$. If the characteristic is 2, then $\chi_M(\lambda) = (\lambda - 1)^4$. Since, $(M - I)^3 \neq 0$ we conclude, again, that $\chi_M = \mu_M$.

(b) A matrix over a field F is diagonalizable if and only its minimimal polynomial in F splits in F and has distinct roots. Here, $\mu_M(\lambda)$, since it has complex roots, does not split in $\mathbb{R}$, hence M is not diagonalizable.

(c) In part (a) we established that μ_M has distinct roots over both $\mathbb{C}$ and $\mathbb{Z}/5\mathbb{Z}$. So in order to determine whether M is diagonalizable we need to determine whether μ_M splits. In $\mathbb{C}$ every polynomial splits, so M is diagonalizable. Finding the roots of μ_M yields

$$D_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Suppose $F = \mathbb{Z}/5\mathbb{Z}$, since U_5, F's multiplicative group, has order 4 we obtain that $\mu_M(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$. Therefore,

$$D_F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

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Problem 12. *Let V be the vector space of C^∞ real-valued functions on $\mathbb{R}$. Consider the following maps $T_i : V \to V$.*

$$T_1(f) = f'' - 6f' + 9f$$
$$T_2(f) = f' - xf$$
$$T_3(f) = ff'$$

(a) Which of the maps T_i are linear transformations?
(b) For each one that is, find a basis for the kernel.

Solution. (a) First note, that $D : C^\infty \to C^\infty$, defined by $D(f) = f'$ is linear. Therefore, the operators T_1 and T_2 are linear as a linear sum of linear operators.

The operator T_3 is not a linear operator. Indeed, take $f = x$. Then, $T_3(2f) = 4x$ and $2T_3(f) = 2x$, hence $T_3(2f) \neq 2T_3(f)$.

(b) To find the kernel for T_1 we need to solve the homogeneous ODE

$$y'' - 6y' + 9y = 0.$$

The characteristic polynomial is $r^2 - 6r + 9 = (r - 3)^2$. Therefore, a basis for the kernel is e^{3x}, xe^{3x}.

For T_2 we have the ODE

$$y' - xy = 0.$$

The integrating factor is $M(x) = e^{-\frac{x^2}{2}}$, therefore the general solution is given by $y = ce^{\frac{x^2}{2}}$. So, we can pick $v = e^{\frac{x^2}{2}}$ as the basis vector.

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