Math 110

Fall 2014

Final Exam

Solutions

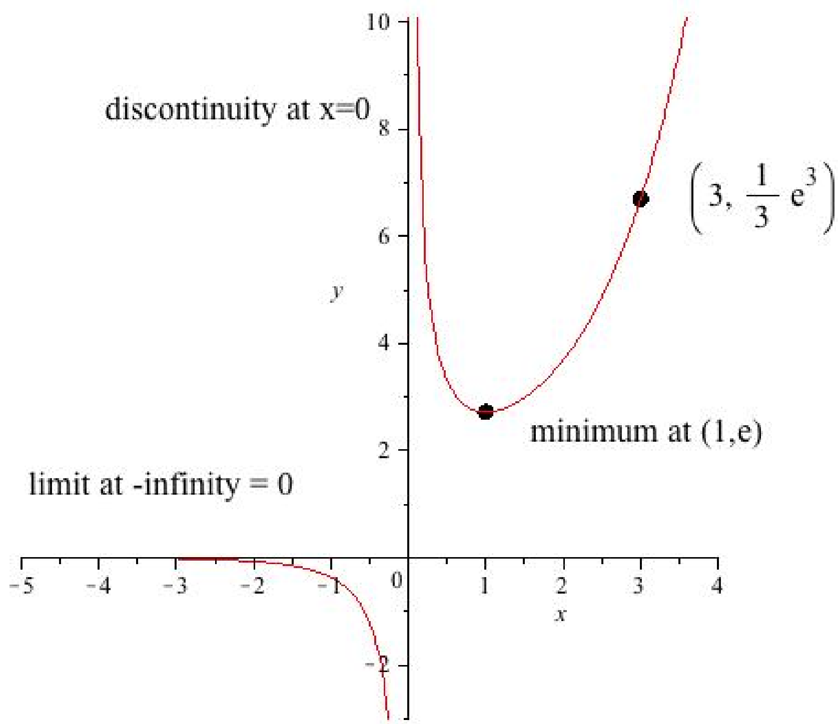
1. Graph these functions. The main thing is to get the general shape, but please also mark a few points and any maxima, minima, asymptotes or discontinuities.

   (a) $y = \dfrac{e^x}{x}$

   Solution: The limit at $-\infty$ is zero (use L'Hôpital's rule). The x in the deniminator produces a vertical asymptote at $x = 0$, with the function positive to the right and negative to the left. There is just one minimum, which may be found by diferentiating.

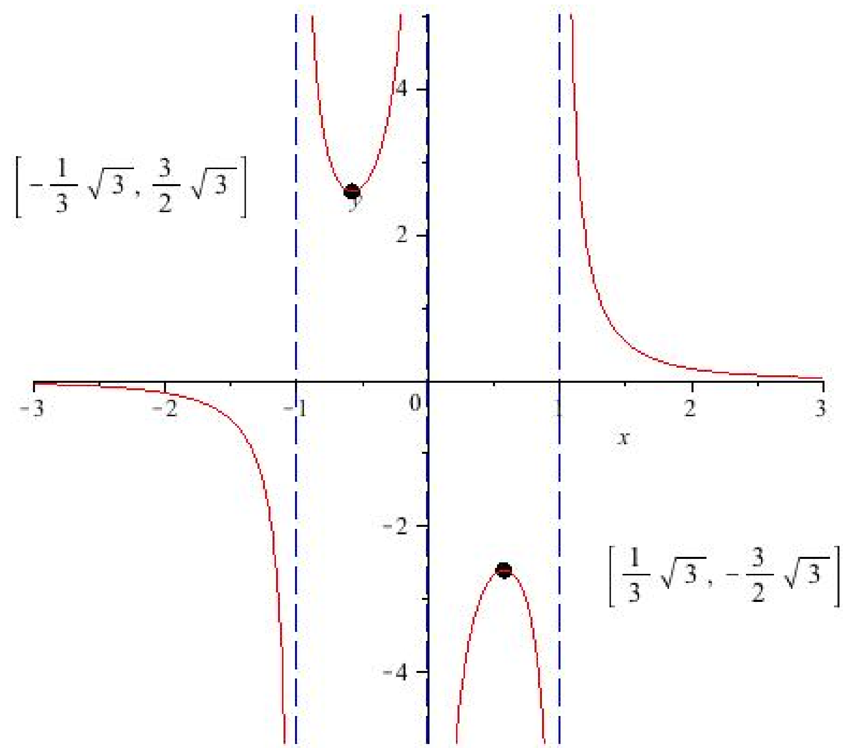
$$\frac{d}{dx}\frac{e^x}{x} = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

   which vanishes at $x = 1$.



(b) $y = \dfrac{1}{x^3 - x}$

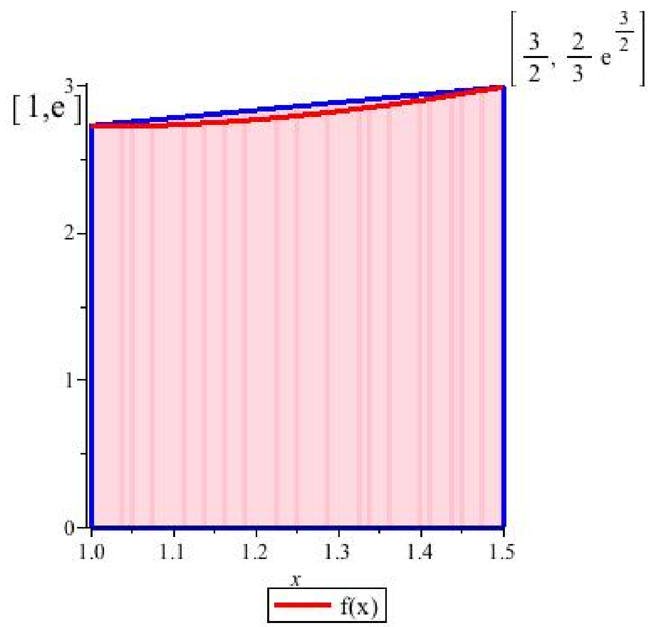
Solution: The limit at $\pm\infty$ is zero. The vertical asymptotes are where the denominator vanishes. Factoring, $x(x+1)(x-1) = 0$ at $-1, 0$ and 1. Testing the signs just to the right and left of the asymptotes gives the right shape. The denominator has maxima and minima in the same locations as the function. Differentiang $x^3 - x$ and setting it equal to zero gives $x = \pm\sqrt{1/3}$, leading to the two points plotted.



2. Compute is the trapezoidal approximation to $\int_1^{3/2} \frac{e^x}{x}\,dx$ using just one trapezoid. Leave the expression in exact form here.

   The one trapezoid has base 1/2 and heights e^1 and $\frac{2}{3}e^{3/2}$. Therefore the trapezoidal approximation to the integral is

$$\frac{1}{4}\left(e + \frac{2}{3}e^{3/2}\right).$$



Then select which of these is closest to the value you have written down.

(i) 0.3

(ii) 0.5

(iii) 1.0

(iv) 1.4

(v) 2.0

To evaluate numerically, we can use $e \approx 2.7$ but then we must come up with a reasonable value for $e^{3/2}$. Here are two ways to do that. (1) $e^{3/2} \approx 10^{1.5/2.3} \approx 10^{0.65}$ which should be about midway between $10^{0.6} \approx 4$ and $10^{0.7} \approx 5$. So we say $e^{3/2} \approx 4.5$. (2) Use the Taylor series: $e^{3/2} = e * e^{1/2} \approx e * (1 + 1/2 + (1/2)^2/2) = e * 13/8 \approx 2.7 * 13/8 = 35.1/8 \approx 4.5$. In either case, we get about $\frac{1}{4}\left(2.7 + \frac{2}{3}4.5\right) = \frac{5.7}{4}$ = a little bigger than 1.4, so we choose answer (iv).

3. (a) Use linearization to estimate $\log_{10} 1.069$.

Solution: $\dfrac{d}{dx}\log_{10} x$ at $x = 1$ is equal to $\dfrac{1}{\ln 10}$, therefore $L(x) = \dfrac{1}{\ln 10}(x-1)$. Plugging in $x = 1.069$ gives

$$\log_{10} 1.069 \approx \frac{1}{\ln 10}(1.069 - 1) \approx \frac{0.069}{2.3} = 0.03\,.$$

(b) Use this to estimate $\log_{10}\left(1.069^{30}\right)$.

Solution: $\log_{10}\left(1.069^{30}\right) = 30\log_{10} 1.069 \approx 30 \cdot 0.03 = 0.9$.

(c) Use this to estimate 1.069^{30}.

Solution: In the middle we use the fact that $10^{0.3}$ is approximately 2 (it's on the log cheatsheet):

$$1.069^{30} = 10^{\log_{10}(1.069^{30})} \approx 10^{0.9} = \left(10^{0.3}\right)^3 \approx 2^3 = 8\,.$$

4. Give the general solution of the differential equation $(1+t)y' + y = \sqrt{t}$.

Solution: Dividing by $1+t$ puts this in the form $y' + P(t)y = Q(t)$ with $P(t) = 1/(1+t)$ and $Q(t) = \sqrt{t}/(1+t)$. The antiderivative of $P(t)$ is $\ln(1+t)$ so the integrating factor is $v(t) = e^{\int P(t)} = e^{\ln(1+t)} = 1+t$. We are supposed to multiply by this which puts us back where we started, only now we know that the left-hand side integrates to $(1+t)y$. Integrating both sides of the equation gives

$$(1+t)y = \frac{2}{3}t^{3/2} + c$$

and therefore

$$y(t) = \frac{2t^{3/2} + C}{3(t+1)}$$

where C is any real number. We can also write this as

$$y(t) = \frac{2}{3}\frac{t^{3/2}}{t+1} + \frac{c}{t+1}$$

where c is any real number.

5. Uncle Sam has a debt of 18 trillion dollars (as of January 1, 2015). Senator Paul decides he needs to pay it off, and proposes an installment scheme, under which Uncle Sam will make payments of one trillion dollars on December 31, 2015 and every December 31 thereafter. Unfortunately, the debt increases by 5% during the course of every year.

   (a) Write expressions for the amount owed by Uncle Sam on January 1 of 2015, 2016 and 2017.

   Solution: Each year the new debt is the old debt multiplied by 1.05, with one trillion subtracted. This give the following table.

date	debt in trillions of dollars
Jan 1, 2015	18
Jan 1, 2016	$18 \times 1.05 - 1$
Jan 1, 2015	$18 \times 1.05^2 - 1.05 - 1$

   (b) Write an expression using Sigma notation for how much Uncle Sam owes on January 1 of year n, counting 2015 as year 0.

   Solution: After year n, Uncle Sam's debt is given by

$$18 \times 1.05^n - \sum_{k=0}^{n-1} 1.05^k \,.$$

(c) Evaluate the sum to get an algebraic expression.

Solution: The geometric series has first term 1 and $r = 1.05$. Plugging into the formula for a geometric series gives

$$\sum_{k=0}^{n-1} 1.05^k = \frac{1.05^n - 1}{1.05 - 1} = 20\left(1.05^n - 1\right) .$$

Therefore, on January 1 of year n, Uncle Sam owes

$$18 \times 1.05^n - 20\left(1.05^n - 1\right) \;=\; 20 - 2 \times 1.05^n \text{ trillion dollars.}$$

(d) Solve for the number of years, n, in which the debt will be paid off. Please leave this as an exact expression (it is OK if this leads to a value which is not a whole number).

Solution: From $20 - 2 \cdot 1.05^n = 0$ we get $1.05^n = 10$. We can write this as

$$n = \log_{1.05} 10 .$$

It may be somewhat easier to continue to simplify. An alternative expression is

$$n = \frac{\ln 10}{\ln 1.05} .$$

(e) Estimate the numerical value of n to the nearest whole number.

Solution: Estimate the value of $\ln 1.05$ by the linearization of $\ln x$ near 1 whicih is $x - 1$. Thus $\ln 1.05 \approx 0.05$. Using the log cheatsheet, $\ln 10 \approx 2.3$, therefore

$$\frac{\ln 10}{\ln 1.05} \approx \frac{2.3}{0.5} = 46 .$$

In other words, it will take Uncle Sam about 46 years to pay off this debt according to Senator Paul's schedule.

6. Let $f(x) = \int_1^x \frac{e^x}{x}\, dx$.

(a) Compute the linear and quadratic Taylor polynomials for f about the point $x = 1$.

Solution: By the fundamential Theorem of Calculus, $f'(x) = e^x/x$. Differentiating again, $f''(x) = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$. To compute the Taylor series around $a = 1$, evaluate f, f' and f'' at 1 to get

$$\begin{aligned} f(1) &= 0 \\ f'(1) &= e \\ f''(1) &= 0\,. \end{aligned}$$

Plug into the formula for Taylor polynomials. This yields identical expressions for P_1 and P_2 (which happens whenver $f''(a) = 0$).

$$\begin{aligned} L(x) &= e(x-1) \\ P_2(x) &= e(x-1) \end{aligned}$$

(b) Use these to estimate $f(3/2)$. Leave as exact expresssions; do not evaluate numerically.

Solution: Plugging in $x = 3/2$ gives

$$f(3/2) \approx \frac{e}{2}\,.$$

Although the problem does not ask for numerics, we note that this is a little over 1.35 because e is a little over 2.7.

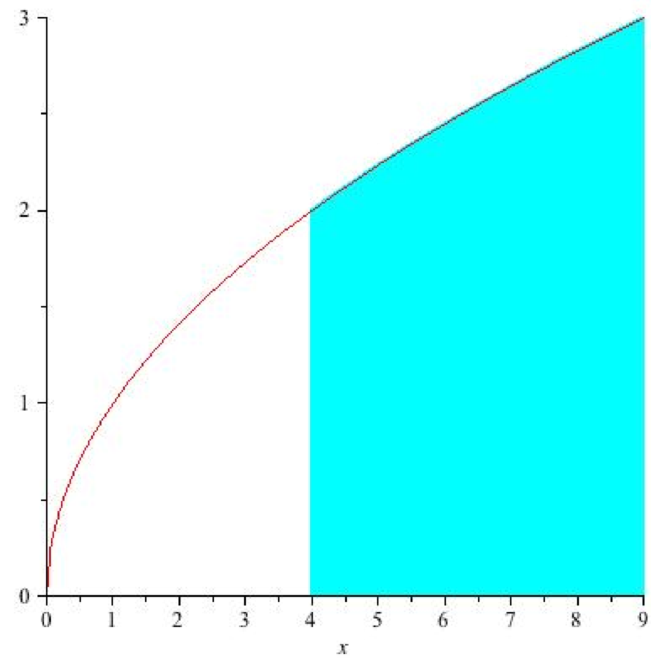
(c) Is the linear estimate an over- or under-estimate of the true value of $f(3/2)$?

Solution: The function e^x is concave upward on $[1, 3/2]$ because the second derivative $e^x(1/x - 1/x^2)$ is positive (zero at the left endpoint, strictly positive everywhere else). Therefore $L(x)$ is an under estimate of the true value of $f(x)$. In fact, we saw in Problem 2 that the trapezoidal method gives an estimate of about 1.4, which is in fact bigger than 1.35.

7. Sketch the region and evaluate the integral.

$$\int_4^9 \int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy\, dx$$

Solution:



The indefinite integral of $e^{y/\sqrt{x}}\, dy$ may be evaluated by the substitution $u = y/\sqrt{x}$; it is equal to $\sqrt{x}e^{y/\sqrt{x}}$. Therefore the inner integral comes out to

$$\int_0^{\sqrt{x}} e^{y/\sqrt{x}}\, dy = \sqrt{x}e^{y/\sqrt{x}}\Big|_{y=0}^{\sqrt{x}} = \sqrt{x}(e-1)\,.$$

The outer integral then evaluates to

$$\int_4^9 \sqrt{x}(e-1)\, dx = (e-1)\,\frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{3}(27-8)(e-1) = \frac{38}{3}(e-1).$$

8. The price of a turkey is proportional to its weight and inversely proportional to the square of its age. Jack's mother gives him enough money to buy a 10 pound turkey that is one year old. When Jack gets to the fair, he realizes that he needs a turkey that is slightly bigger. How much older will the turkey have to be per extra pound in weight?

   Please begin by writing down an equation satisfied by price, weight and age, giving the interpretation and units for all varialbles and constants used.

Solution: The wording of the problem may be translated into the equation

$$p = k\frac{w}{a^2}$$

where w is the weight (the problem suggests units of pounds), a is its age (units of years), and p is the price of the turkey (units of in dollars or your favorite medieval currency). The proportionality constant k then has units of dollars times years squared per pound.

The problem calls for us to evaluate da/dw along the level curve where p is constant, throught the point $(w, a) = (10, 1)$. We use the formula $\frac{da}{dw} = -\frac{p_w}{p_a}$. The partial derivatives of $p(w, a)$ are given by

$$\begin{aligned}
\frac{\partial p}{\partial w}(w,a) &= \frac{k}{a^2} \\
\frac{\partial p}{\partial a}(w,a) &= \frac{-2kw}{a^3}\,.
\end{aligned}$$

Therefore,

$$\frac{da}{dw} = -\frac{p_w}{p_a} = \frac{k/a^2}{2kw/a^3} = \frac{a}{2w}\,.$$

Evaluating at the point $(10, 1)$ gives $1/20$. Thus, for each pound more tha Jack needs, he must be willing to accept a turkey $1/20$ year older and tougher.

[Note that the units are in years per pound, which is consistent with computation we are trying to compute how mthat it represents how many more years the turkey will have per ound of extra weight.]

9. (a) Compute the gradient of the function $f(x,y) = \dfrac{x^2}{8\sqrt{y}}$.

Solution:

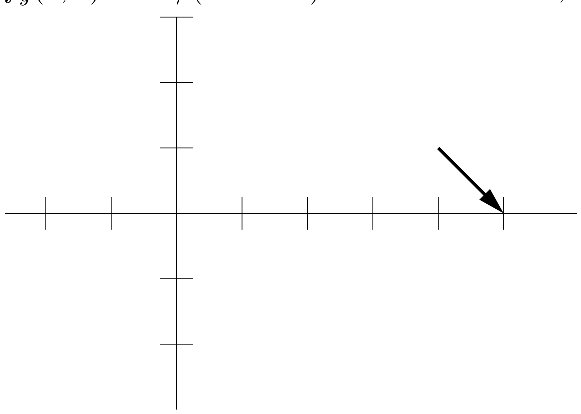
$$\frac{\partial f}{\partial x} = \frac{2x}{8\sqrt{y}} = \frac{x}{4\sqrt{y}}$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{-16y^{3/2}}\,.$$

Therefore the gradient of f is $\dfrac{x}{4\sqrt{y}}\hat{\mathbf{i}} - \dfrac{x^2}{16y^{3/2}}\hat{\mathbf{j}}$.

(b) Evaluate the gradient of f at the point $(4,1)$ and draw this vector on these coordinate axes starting at the point $(4,1)$.

Solution: Evaluating at $(4,1)$ gives $f_x(4,1) = 4/(4\sqrt{1}) = 1$ and $f_y(4,1) = 16/(-161^{3/2}) = -1$. Therefore, $\nabla f(4,1) = \hat{\mathbf{i}} - \hat{\mathbf{j}}$.



(c) From the point $(4,1)$, which direction should you move in order to increase f the fastest. State this direction by giving a unit vector pointing in the direction.

Solution: The fastest increase is in the direction of the gradient. A unit vector in this direction is obtained by dividing the gradient by its length, yielding $\mathbf{u} = \dfrac{1}{\sqrt{2}}\hat{\mathbf{i}} - \dfrac{1}{\sqrt{2}}\hat{\mathbf{j}}$.

(d) At the point $(4,1)$, how fast does f increase per unit moved in this direction?

Solution: The rate of change of f in this direction is $\nabla f \cdot u$ which is always equal to $|\nabla f|$ when u is a unit vector parallel to ∇f (use the dot product identity). Thus, the rate of increase of f in the maximal direction is $|\nabla f(4,1)| = \sqrt{2}$.

10. Uncle Sam owes 18 trillion dollars (this is a new problem, really!). Interest accmulates continuously at the rate of 5%/year. Suppose that Congress forces Uncle Sam to pay back the debt continuously at a rate of one trillion dollars per year.

(a) Write a differential equation for the amount owed by Uncle Sam at time t. Please give the meaning and units of all variables.

Solution: Let $A(t)$ be the amount owed at time t, measured in trillions of dollars. Then

$$\frac{dA(t)}{dt} = 0.05A(t) - 1$$

where the constant 0.05 has units of inverse years and the constant 1 has units of trillions of dollars per year.

(b) Find the general solution of this differential equation.

Solution: The equation $A' - 0.05A = -1$ is first order linear with P and Q both constant: $P(t) = -0.05$ and $Q(t) = -1$. Multiplying by the integrating factor $e^{-0.05t}$ and integrating yields

$$e^{-0.05t}A(t) = 20e^{-0.05t} + C$$

and therefore

$$A(t) = 20 + Ce^{0.05t}\,.$$

(c) State the initial condition and give the solution to the initial value problem.

Solution: The initial condition is $A(0) = 18$. Solving for C gives $C = -2$ therefore $A(t) = 20 - 2e^{t/20}$.

(d) How long will it take for Uncle Sam's debt to reach zero?

Solution: Setting $A(t) = 0$ gives $e^{t/20} = 10$ hence $t = 20 \ln 10$. Numerically, this comes out to about 46 years. [Note: this is similar to Problem 4 except that this version does not require you to approximate $\ln 1.05$.]

11. (a) Compute the indefinite integral $\int x^{-3} \ln x \, dx$.

Solution: This is a straightforward integration by parts. Let $u = \ln x$ and $dv = x^{-3} dx$ so that $v = -(1/2)x^{-2}\, dx$ and $dv = x^{-1}\, dx$. Then

$$\int x^{-3} \ln x \, dx = -\frac{1}{2}x^{-2} \ln x + \frac{1}{2}\int x^{-2}(x^{-1}\, dx) = -\frac{1}{2}x^{-2} \ln x - \frac{1}{4}x^{-2}.$$

Alternative solution: You can do this as a substitution with $u = \ln x$ and $du = \frac{dx}{x}$. The integral then becomes $\int x^{-2} \ln x (dx/x) = \int e^{-2u} u \, du$. Now you can integrate by parts to get

$$\int ue^{-2u}\, du = \frac{-1}{2}ue^{-2u} + \int \frac{e^{-2u}}{2}\, du = \frac{-1}{2}ue^{-2u} - \frac{1}{4}e^{-2u}$$

and plugging in $u = \ln x$ produces the same answer as before.

(b) Write the improper integral $\int_1^\infty x^{-3} \ln x \, dx$ as a limit and evaluate it.

Solution: The integrand is continous so it is a type-I improper integral. Using the previous result,

$$\begin{aligned}
\int_1^\infty x^{-3} \ln x \, dx &= \lim_{M\to\infty} \int_1^M x^{-3} \ln x \, dx \\
&= \lim_{M\to\infty} \left(-\frac{1}{2}x^{-2} \ln x - \frac{1}{4}x^{-2} \right)\Bigg|_1^M \\
&= \lim_{M\to\infty} \left(-\frac{1}{2}M^{-2} \ln M - \frac{1}{4}M^{-2} \right) + \frac{1}{4} \\
&= \frac{1}{4}\,.
\end{aligned}$$

It is obvious that the term $(1/4)M^{-2}$ goes to zero; to see why $(1/2)M^{-2} \ln M$ also goes to zero note that $\ln M$ grows much slower than any power of M.

12. Suppose that $f(x, y)$ is a function and that x and y depend on parameters s and t by the formulas $x = \sqrt{s^2 + t}$ and $y = \ln(t - 4)$. Which of the following expressions correctly describes the rate of change of f with respect to t when (s, t) starts at the value $(2, 5)$ and then t is varied while s is held constant? You need only circle the correct number from (i) to (v).

(i) $\dfrac{2}{3}\dfrac{\partial f}{\partial x}(2, 5) + \dfrac{\partial f}{\partial y}(2, 5)$

(ii) $\dfrac{2}{3}\dfrac{\partial f}{\partial x}(3, 0) + \dfrac{\partial f}{\partial y}(3, 0)$

(iii) $\dfrac{1}{6}\dfrac{\partial f}{\partial x}(2, 5) + \dfrac{\partial f}{\partial y}(2, 5)$

(iv) $\dfrac{1}{6}\dfrac{\partial f}{\partial x}(3, 0) + \dfrac{\partial f}{\partial y}(3, 0)$

(v) None of the above

Solution: When $(s, t) = (2, 5)$ the value of (x, y) is given by $(\sqrt{2^2 + 5}, \ln(5 - 4)) = (3, 0)$. The function f is not known to us so its partial derivatives of $(x(t), y(t))$ are known only as $\dfrac{\partial f}{\partial x}(3, 0)$ and $\dfrac{\partial f}{\partial y}(3, 0)$ respectively. [Note that the partial derivative of f in terms of x and y is evaluated at (x, y), not at (u, v).] We do know explicitly that

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{1}{2\sqrt{s^2 + t}} \\ \frac{\partial y}{\partial t} &= \frac{1}{t - 4}. \end{aligned}$$

Evaluating at $(2, 5)$ gives $1/6$ and 1 respectively. The chain rule then gives (iv).

13. A swimming pool holding 300 cubic meters of water is determined to contain 1/100 of a cubic meter of toxins. Immediately a drain is opened and pool water starts flowing out at 3 cubic meters per minute. Also a hose is inserted to pump in fresh water at the rate of 1 cubic meter per minute. Assuming the fresh water mixes rapidly with the pool water, which of these differential equations models the total amount P of poison in the pool at time t minutes after the toxicity is discovered? You need only circle the correct number from (i) to (v).

(i) $P'(t) = -\dfrac{3P(t)}{300 - 2t}$

(ii) $P'(t) = -1 + \dfrac{P(t)}{3}$

(iii) $\dfrac{P'(t)}{P(t)} = -\dfrac{1/100}{300}$

(iv) $P'(t) - P(t) = -\dfrac{2}{300 - t}$

(v) None of the above

Solution: The pool is filling at rate 1 $m^3/\min$ and draining at rate 3 $m^3/\min$. Therefore, the amount of water in the pool at time t is $300 - 2t$ cubic meters. If $P(t)$ is the total amount of poison in the pool, the concentration of poison is $P(t)/(300-2t)$. The amount of poison leaving the pool is the concentration times the total outflow, which is $3m^3/\min$ times the concentration. Because no poison enters the pool, this is the rate of change of $P(t)$ and it is negative. Thus the correct answer is (i).

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \\ \log_{10}(3) &\approx 0.48 \end{aligned}$$

Some other useful quantities to with 1%:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \end{aligned}$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$

21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (a >$

22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \qquad (x >$