Name: ______________________________ Section (circle one): 001 002

# Midterm Exam I for Math 110, Fall 2017

## Solutions

October 03, 2017

| Problem | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 9 | |
| 4 | 9 | |
| 5 | 15 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 15 | |
| 9 | 6 | |
| 10 | 9 | |
| 11 | 9 | |
| Total | 120 | |

- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One $8.5 \times 11$ cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. Suppose $x$ and $y$ are related by $\ln x = 3 \ln y$. State in words the relation between $x$ and $y$.

**Solution:** Exponentiate both sides with base $e$: $x = e^{3 \ln y} = (e^{\ln y})^3 = y^3$

In words, $x$ is equal to the cube of $y$, or $y$ is equal to the cube root of $x$ (either one of these is fine).

2. Let $f(x) = \dfrac{\sin x}{x}$.

(a) Compute $\lim_{x\to 0} f(x)$.

**Solution:** Both the numerator and denominator are zero at $x = 0$ therefore we may apply L'Hopital's rule. This gives

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos(0)}{1} = 1\,.$$

(b) Compute $f'(x)$.

**Solution:** By the quotient rule: $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$.

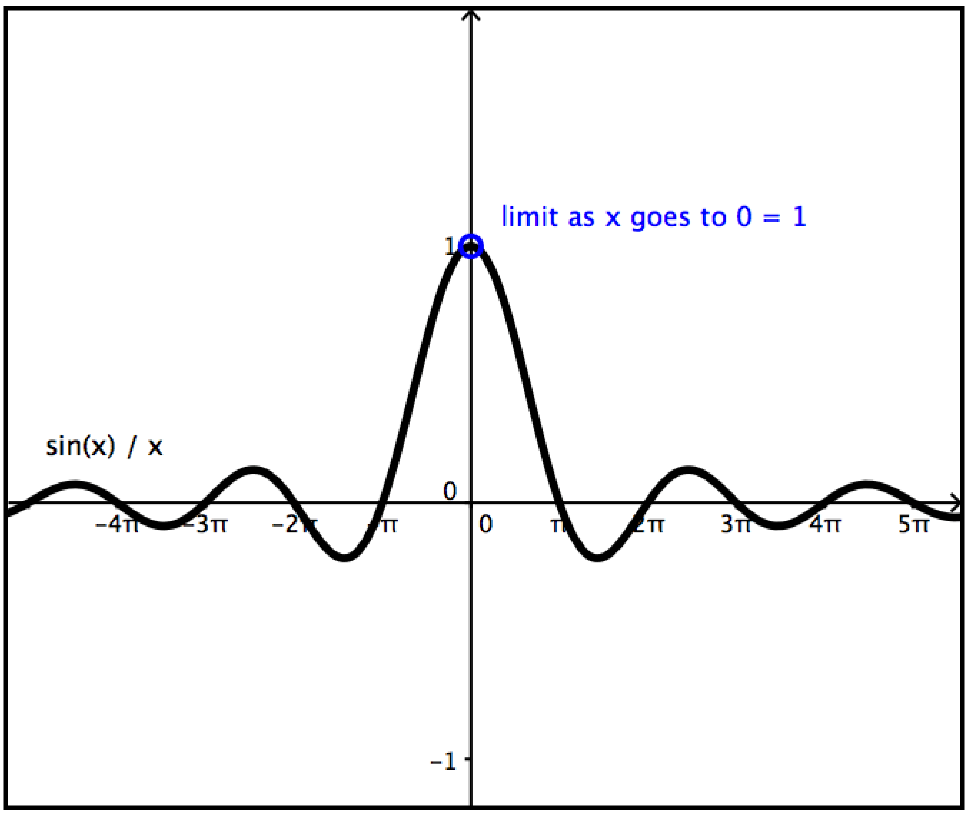
(c) Compute $\lim_{x\to 0} f'(x)$.

**Solution:** Again, numerator and denominator are zero at zero and we may apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^2} = \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x} = \lim_{x\to 0} -\frac{\sin x}{2} = 0\,.$$

(d) Sketch a graph of the function $f$. Your graph should reflect what you computed above, as well as qualitative features of the function as $x \to \pm\infty$.

**Solution:** Use your answers to parts (a) and (c), you can see that although the graph has a hole at $x = 0$, the function approaches 1 there and the derivative approaches zero there. So there's a hole at $(0, 1)$ and the function is flat there. It's not too hard to see it's a maximum, so this gives the shape near $(0, 1)$. Everywhere else the function oscillates like sin, having the same zeros but with the oscillations getting smaller away from the $y$-axis.



3. Write an equation for the following scenario. Be sure to give an interpretation for all variables and constants and to give units for each.

   > The amount of energy per time required to maintain a refrigeration unit starts at a level proportional to the volume of the unit and decreases exponentially after that.

   **Solution:** $P = aVe^{-bt}$ where

   - $P$ is in units of energy per time, also known as power, for example Watts; it represents what was asked, "the amount of energy per time…"
   - $a$ is a constant of proportionality with units of power per volume or Watts per cubic meter (or cubic centimter, etc.).
   - $V$ is in units of volume and represents the volume of the fridge.
   - $b$ is a time constant in units of inverse time, representing the reciprocal of the time to fall to $1/e$ of it's original value.

   Note: the constant $b$ is positive if you wrote the formula as $aVe^{-bt}$, but negative if you wrote the formula as $aVe^{bt}$.

4. Find a function $g(x) = cx^p$ such that $f(x) \sim g(x)$ as $x \to \infty$ where

$$f(x) = \frac{\sqrt{4/x}}{\frac{3}{x} + \frac{5}{x^2}}.$$

You do not have to prove your answer but we should be able to see how you got it.

**Solution:** Because $5/x^2 \ll 3/x$ we can ignore this term when the two are added. That yields

$$f(x) \sim \frac{2x^{-1/2}}{3x^{-1}} = \frac{2}{3}x^{1/2}.$$

**Alternate Solution:** Clearing the fractions in the denominator by multiplying top and bottom initially by $x^2$ gives

$$f(x) = \frac{x^2\sqrt{4}\sqrt{x}}{3x+5}.$$

Evaluating $\sqrt{4} = 2$ and ignoring the 5 in the denominator (because $5 \ll 3x$ as $x \to \infty$) gives

$$f(x) \sim \frac{2x^2x^{-1/2}}{3x} = \frac{2}{3}x^{1/2}.$$

5. Louie agree to pay Sam \$50,000 every year for 15 years, starting January 1, 2018, after which Louie gets possession of Sam's condo. Every year, Sam puts the money in an investment account that increases by 6.9% during the course of the year.

   (a) At the end of the day on January 1, 2018, how much money is in the account?

   **Solution:** \$50,000 (unless you explain why this account has some magical property to make it instantly increase).

   (b) At the end of the day on January 1, 2019, how much money is in the account?

   **Solution:** $\$50,000 * 1.069 + \$50,000$

   (c) Let $M$ be the amount of money in the account at the end of the day on January 1, 2032, right after Sam receives the last payment. Write a summation formula for $M$.

   **Solution:** In dollars, one way to write it is $M = \sum_{k=0}^{14} 50000(1.069)^k$ .

(d) Write a formula for $M$ that has no summation in it.

**Solution:** $M$ is the sum of a geometric series with first term $A = 50000$, ratio $r = 1.069$, and number of terms $n = 15$ (sum runs from zero to 14). Therefore,

$$M = 50,000\frac{1.069^{15} - 1}{1.069 - 1}\,.$$

(e) Use your log cheatsheet, linearization, and any other techniques you can think of to give a numerical estimate for the value of $M$. Within 10% is accurate enough.

**Solution:** The only estimation needed is the value of $y = 1.069^{15}$. Taking natural log of both sides gives $\ln y = 15 \ln 1.069$. Use linearization to estimate $\ln(1 + 0.069)$ as $L(1.069) = \ln 1 + 1 * (1.069 - 1) = 0.069$. Then $\ln y$ is about $15 * 0.069 = 1.035$ so $y = 1.069^{15}$ is about $e^{1.035}$. You can use linearization again to estimate $e^{1.035}$ as $L(1.035) = e + e(1.035 - 1)$ or about 2.8, or just use the approximation that something a little larger than $e$ should be more than 2.7 but less than 3. So $M$ is about $\frac{50000(1.8)}{0.069} = \$1,300,000$.

**Alternate Solution:** This time we estimate $1.069^{15}$ by estimating $\log_{10}(1.069^{15}) = 15\log_{10} 1.069$ and estimating this by the linearization for log)10:

$$\log_{10}(1 + 0.069) \approx \log_{10} 1 + (1/\ln 10) * 0.069 \approx 0.069/2.3 = 0.03\,.$$

Therefore

$$1.069^{15} \approx 10^{0.45} \approx 10^{(3/2)\log_{10} 2} = 2^{3/2} = 2\sqrt{2} \approx 2.8$$

and you can proceed as before.

6. In each case, write the sum (call is $S$) in $\Sigma$ notation, then evaluate it (leave in exact form, do not use decimal approximations).

   (a) $e^a + e^{2a} + e^{3a} + \cdots + e^{13a}$

   **Solution:** The formula for summing a finite geometric series gives

$$S = \sum_{k=1}^{13} e^{ak} = \frac{e^a(1-e^{13a})}{1-e^a} = \frac{e^a - e^{14a}}{1-e^a}.$$

   (b) $\frac{2y+z}{x} + \frac{2y+2z}{x} + \cdots + \frac{2y+14z}{x}$

   **Solution:** The formula for summing a finite arithmetic series gives

$$S = \sum_{k=1}^{14} \frac{2y+kz}{x} = 7 * \frac{4y+15z}{x}.$$

   (c) The infinite sum $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

   **Solution:** This is an infinite series with ratio $1/2$. Beacuse $|1/2| < 1$, this is summable. The formula gives

$$S = \sum_{k=0}^{\infty} \frac{1}{2}(\frac{1}{4})^k = \frac{1/2}{1-1/4} = \frac{2}{3}.$$

7. Use integrals to give upper and lower bounds for this sum.

$$\sum_{m=16}^{99} m\sqrt{m}\,.$$

These should be given as analytic expressions, with no integrals left unevaluated, simplified when possible. Please include a sketch on the facing page (back of problem 6) or an extra piece of paper.

**Solution:** Upper bound:

$$\int_{16}^{100} x\sqrt{x}\;dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_{16}^{100} = \frac{2}{5}(10^5 - 4^5) = 40,000 - 409.6 \approx 39,600.$$

**Solution:** Lower bound:

$$\int_{15}^{99} x\sqrt{x}\;dx = \frac{2}{5}x^{\frac{5}{2}}\Big|_{15}^{99} = \frac{2}{5}(99^{\frac{5}{2}} - 15^{\frac{5}{2}})\,.$$

PUT IN FIGURE HERE

Extra credit [2 points] One of these bounds does not simplify to a rational number. Use linearization to give an approximate numerical value for this expression.

**Solution:** Linearization approximates $f(x) = x^{5/2}$ by

$$L(x) = a^{5/2} + (x-1)f'(a) = a^{5/2} + (5/2)a^{3/2}\,.$$

Therefore in the lower bound, we can approximate

$$\begin{aligned} 99^{5/2} &\approx 10^5 - 2500 \\ 15^{5/2} &\approx 4^5 - 160 \end{aligned}$$

Therefore, the lower bound is less than the upper bound by roughly $(2/5)(2500-160) = 1000 - 64 = 936$, putting the lower bound at roughly 38650.

8. Compute these indefinite integrals. One is a substitution, one is by parts, and one uses both of these techniques.

(a) $\int \ln(1+x)\sqrt{1+x}\,dx$

**Solution:** Let $u = ln(1+x)$ and $dv = \sqrt{1+x}\,dx$, so that $du = \frac{1}{1+x}\,dx$ and $v = \frac{2}{3}(x+1)^{\frac{3}{2}}$.
Then $\int u\ dv = u * v - \int v\ du = \frac{2}{3}(x+1)^{\frac{3}{2}}\ln(x+1) - \frac{2}{3}\int \sqrt{x+1}\ dx$
$= \frac{2}{3}(x+1)^{\frac{3}{2}}\ln(x+1) - \frac{4}{9}(x+1)^{\frac{3}{2}} + c$

(Note: we will not penalize for missing constants)

(b) $\int x^2\sqrt{4+x^3}\,dx$

**Solution:** Let $u = 4 + x^3$ so that $\frac{du}{3} = x^2 dx$. Then the integral becomes $\frac{1}{3}\int \sqrt{u}\ du = \frac{2}{9}u^{\frac{3}{2}} + c = \frac{2}{9}(x^3+4)^{\frac{3}{2}} + c$

(c) $\int \arctan x\,dx$

**Solution:** Look up the derivative of $\arctan x$ to be $\frac{1}{1+x^2}$. Then let $u = \arctan x$ and $dv = dx$ so that $du = \frac{1}{1+x^2}\ dx$ and $v = x$.
By integration by parts, the integral becomes $x\arctan x - \int \frac{x}{1+x^2}\ dx$
Now let $w = x^2 + 1$ so that $\frac{dw}{2} = xdx$.
Then $\int \frac{x}{1+x^2}\ dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + c = \ln\sqrt{w} + c = \ln\sqrt{x^2+1} + c$.
So the answer is $x\arctan x - \ln\sqrt{x^2+1} + c$

9. Let $g(x)$ be the time in hours it takes to drill a one foot diameter shaft $x$ meters deep into a bed of limestone. State an interpretation for the inverse function $g^{-1}$.

   **Solution:** $g^{-1}(t)$ is the length in meters of a shaft with diameter one foot that can be drilled into a bed of limestone in $t$ hours.

10. (a) The function $\int_1^x (\ln(t))^2\,dt$ is a function, $f$, of what variable?

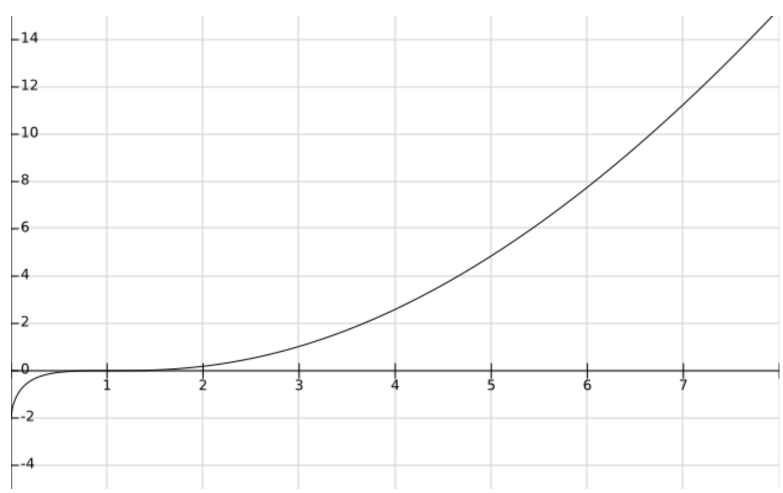
**Solution:** $x$, because $t$ is a bound variable.

(b) What is $f'$?

**Solution:** By the Fundamental Theorem of Calculus, $f'(x) = \ln(x))^2$: the derivative of a definite integral as a function of the upper limit of integration is the integrand evaluated at the upper limit.

(c) Sketch a graph of this function with the input variable ranging from 1 to 7. This can be a very rough sketch; just give an idea of the scale of the vertical axis and whether the function is increasing/decreasing and curved up/down.

**Solution:** The graph should have a point at $(1, 0)$. It should be increasing because its derivative $(\ln x)^2$ is positive and should be curving up because its derivative $(\ln x)^2$ is increasing (or because its second derivative $2(\ln x)/x$ is positive). The value of $(\ln x)^2$ ranges from zero at 1 to about 4 at 7, so integrates to something more than 10 and less than 20. Anything on the order of ten to twenty would be a good choice for the scale of the vertical axis.



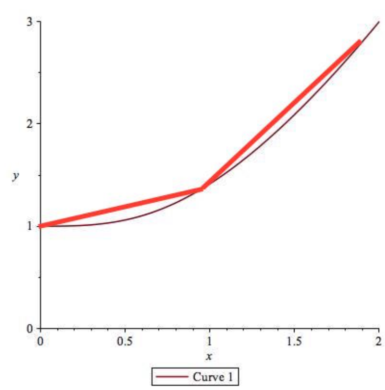
11. Let $A$ be the area under the graph of $y = \sqrt{1+x^3}$, above the $x$-axis, between the $y$-axis and the vertical line $x = 2$. Circle which of these numeric values most closely estimates $A$. If you wish to be considered for partial credit, state your reasoning, give a sketch, etc. [Hint: what kind of approximation will be most accurate if you want the computation to be brief?]

(a) 1.4
(b) 2
(c) 2.25
(d) 2.4
(e) 3.4
(f) 4
(g) 6

**Solution:** 3.4 is the closest. To get an accurate approximation, trapezoids work quite nicely. Even with only 2 trapezoids, you get:

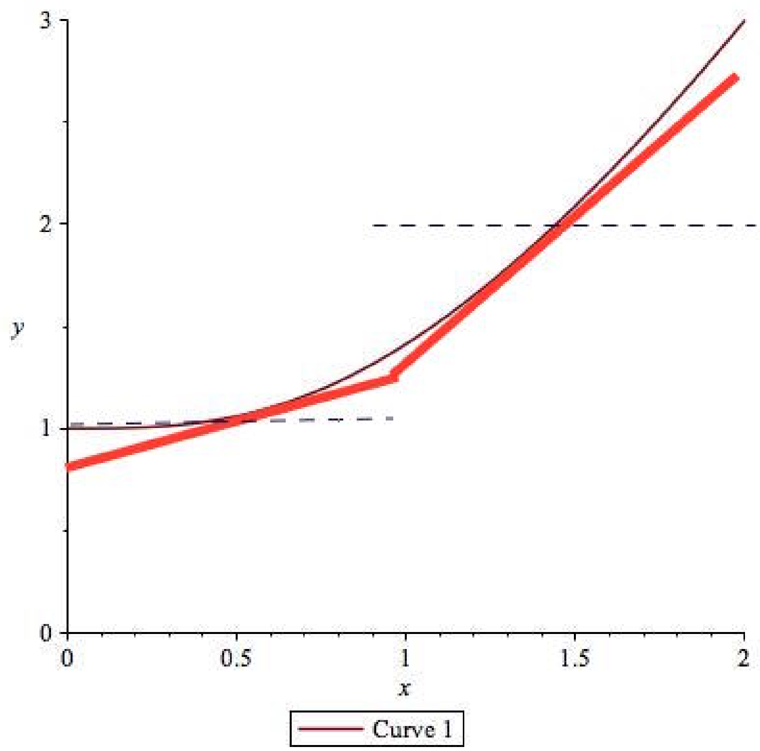
> Bases are each 1; heights of the first trapezoid are 1 and $\sqrt{2}$, heights of the second trapezoid are $\sqrt{2}$ and $\sqrt{1+2^3} = 3$. The approximation is
>
> $$1 * (1 + \sqrt{2})/2) + 1 * (\sqrt{2} + 3)/2 = 2 + \sqrt{2} \approx 3.4\,.$$

The true value of $3.2413\ldots$ is somewhat less because the function curves up and is therefore underneath the chords forming the tops of the trapezoids.



**Note:** If you want to be completely sure the value does not dip below 2.9 (making 2.4 a closer answer), you can try a lower trapezoidal sum with two trapezoids! Divide into two strips of width 1 and make each trapezoid tangent to the graph at the midpoint $x$ value. This gives $y(1/2)+y(3/2)$ as a lower bound, which evalues to $(3/4)\sqrt{2}+(1/4)\sqrt{70} > 4.2/4+8.4/4 > 3$.

We did not expect you to do this! But it does emphasize that finding a bound is a creative process.




The areas of the trapezoids is the same if the slanted segment (red) forming top of each one is replaced by the horizontal line (black) going through the midpoint of said slanted line. This gives an easy calculation because the heights of the rectangles whose tops are the dashed lines are $f(1/2)$ and $f(3/2)$ where $f(x) = \sqrt{1+x^3}$.