Name: ____ Section (circle one): 001 002

Midterm Exam I for Math 110, Fall 2015

Solutions

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	8	
7	16	
8	12	
9	8	
10	16	
Total	120	

- You have ninety minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. What is the (approximate) relation between x and y if $\ln y = 4.6 + 2\ln x$?

Solution: Exponentiating both sides,

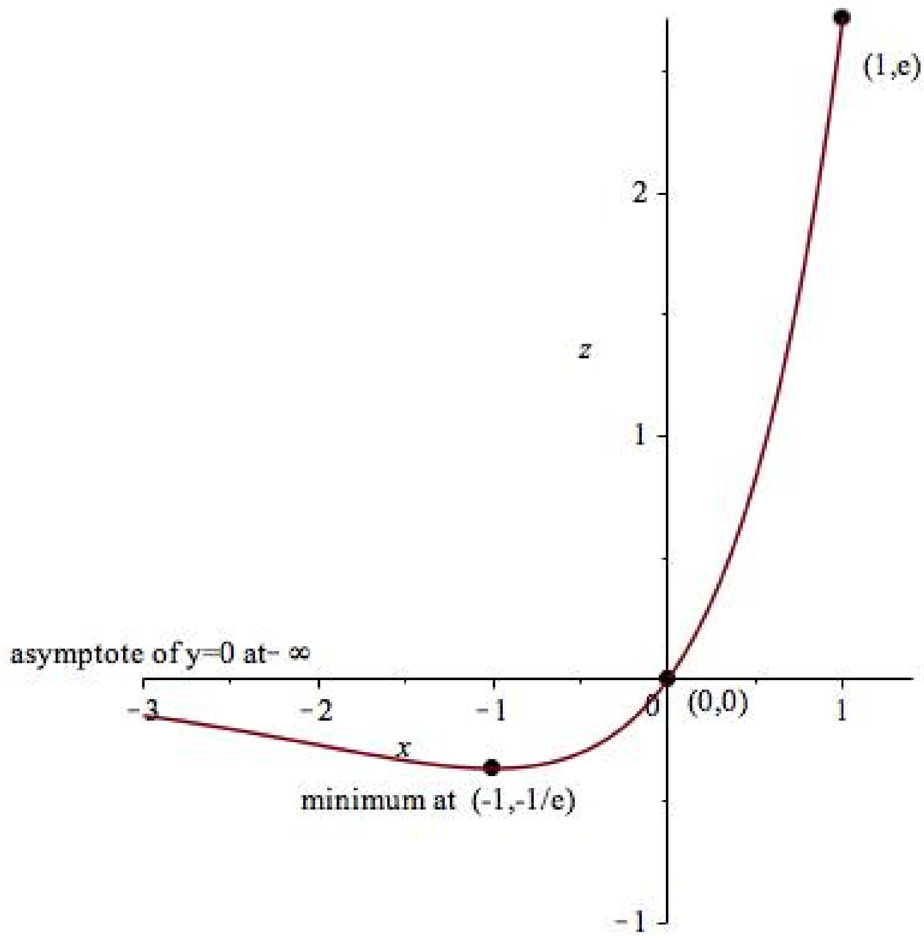
$$y = e^{4.6} \cdot e^{2\ln x} = e^{2.3\times 2} \cdot (e^{\ln x})^2$$

Because $e^{2.3} \approx 10$ we get an approximate value for this of $10^2 x^2 = 100x^2$.

Thus, $\boxed{y \approx 100x^2}$

2. Graph the function xe^x. Please choose scales on the x- and y-axes that are neither too small nor too large to show the shape, and point out any maxima, minima and asymptotes, as well as exact coordinates of a few points on the graph.

Solution:



3. Jake sums the reciprocals of the square roots of the integers from 1 to 100. Laura sums the reciprocals of the square roots from 1 to 200.

(a) Write expressions for Jake's sum J and Laura's sum L in $\sum$ notation.

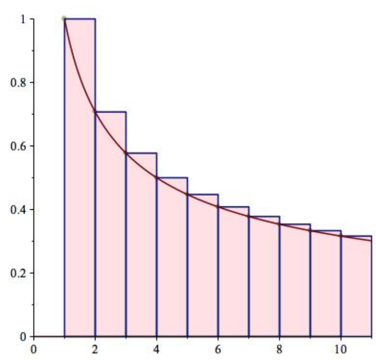
Solution:

$$J = \sum_{n=1}^{100} \frac{1}{\sqrt{n}}$$

$$L = \sum_{n=1}^{200} \frac{1}{\sqrt{n}}$$

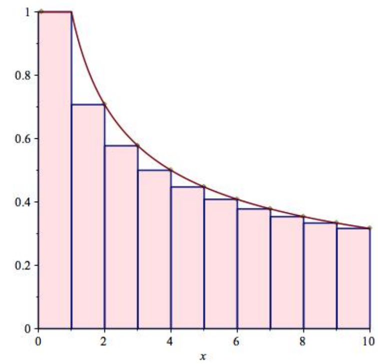
(b) Find a lower bound for J using integrals.

Solution: $J \geq \int_1^{101} x^{-1/2}\, dx = 2(\sqrt{101} - 1) \approx 18.1$ by linear approximation. In the figure, only the first 10 rectangles out of 100 are shown.



(c) Find an upper bound for J using integrals.

Solution: $J \le 1 + \int_1^{100} x^{-1/2}\,dx = 1 + 2(\sqrt{100} - 1) = 19$ as shown by the following figure (again, only ten rectangles shown).



(d) What simple expression approximates the ratio L/J?
The same expression should work if Jake summed the first million reciprocal square roots and Laura summed the first two million.

Solution: $J \approx 2\sqrt{100}$ and $L \approx 2\sqrt{200}$ so $\boxed{L/J \approx \sqrt{2}.}$

4. A retirement account growing at an exponential rate is expected to double in 14 years.

(a) Write an equation for the amount in the account after t years in terms of the starting contribution, assuming no further contributions. Be sure to give units for all constants.

Solution: Let $M(t)$ be the amount in dollars after time t years. If the starting contribution is S dollars, then exponential growth means $M(t) = Se^{\alpha t}$ for some α. The units of α are inverse time. In fact, doubling after 14 years means $14\alpha = \ln 2$ so $\boxed{\alpha = (\ln 2)/14}$ inverse years. (Numerically this is $\approx 0.7/14 = 0.05$ inverse years; we did not ask for numerics on this part of the problem so you don't need this yet.)

(b) Using your log cheatsheet, give an approximate value for the fund after 32 years if it starts with $700,000. This can be very approximate - we only care about the first digit.

Solution:

$$\begin{aligned} M(32) &= 700,000 \times e^{32 \ln 2/14} \\ &\approx 700,000 \times e^{1.6} \text{ (see above)} \\ &\approx 700,000 \times 10^{1.6/2.3} \\ &\approx 700,000 \times 10^{0.7} \\ &\approx 700,000 \times 5 \\ &= 3,500,000 \end{aligned}$$

5. In each case, write the sum (call is S) in Σ notation, then evaluate it (leave in exact form, do not use decimal approximations).

(a) $3Z + (4Z - 13) + (5Z - 26) + \cdots + (13Z - 130)$

Solution: This is an arithmetic series with 11 terms, each term increasing by $Z - 13$. We may write this as

$$\sum_{n=0}^{10} 3Z + n(Z - 13)\,.$$

The average of the terms of an arithmetic series is the average of the first and last term, in this case $\dfrac{3Z + (13Z - 130)}{2} = 8Z - 65$. Therefore the sum is

$$11 \times (8Z - 65) = \boxed{\text{88Z - 715}}\,.$$

(b) The infinite sum $6e^{-2} + 18e^{-4} + 54e^{-6} + \cdots$

Solution: This is an infinite geometric series with ratio $r = 3e^{-2}$ and first term $A = 6e^{-2}$. Using the formula $S = A/(1 - r)$, the infinite sum is

$$\frac{6e^{-2}}{1 - 3e^{-2}}\,.$$

This can be simplified to $\boxed{\dfrac{6}{e^2 - 3}}$ if desired; either is correct.

6. Write an equation expressing the following scenario. Be sure to define variables and constants and to give units for each.

   The number of troops necessary to patrol an area after combat units are withdrawn is proportional to the 1/2 power of the area being patrolled and inversely proportional to the length of time after the withdrawal.

Solution: Let N be the number of troops measured in people. Let A be the patrolled area measured in some squared distance unit such as square miles. Let t be the time since withdrawal measured in some time unit such as days. Then

$$\boxed{N = C\frac{\sqrt{A}}{t} \text{ where } C \text{ has units of people days per mile.}}$$

One could instead give the generic unit: the units of C are people times time divided by length.

7. (a) Compute $\lim_{x\to 0^+} x \ln x$.

Solution: In order to use L'Hôpital's rule, write this is $\ln x/(1/x)$ which is $-\infty/\infty$ as $x \to 0^+$. Then the limit is equal to

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = \boxed{0}.$$

Alternatively one could write the original expression as $x/(1/\ln x)$, but that would lead to a longer computation.

(b) Compute $\lim_{x\to\infty} \frac{\sqrt{2x^4+1}}{3x^2}$.

Solution: Dividing top and bottom by x^2 gives $\lim_{x\to\infty} \frac{\sqrt{2+x^{-4}}}{3}$. As $x \to \infty$, the x^{-4} term goes to zero, leaving the answer $\boxed{\sqrt{2}/3}$.

(c) True or false? $\sqrt{2x^3 - 1} \sim x^{3/2}$ as $x \to \infty$

Solution: $\boxed{\text{FALSE.}}$ For this to be true, we would need $\lim_{x\to\infty} \frac{\sqrt{2x^3-1}}{x^{3/2}} = 1$. But by the same computation as in the previous part, dividing top and bottom by $x^{3/2}$, we get a limit that is equal to $\sqrt{2}$, hence not equal to 1.

(d) True or false? $(\ln x)^2 \ll x$ as $x \to \infty$

Solution: $\boxed{\text{TRUE.}}$ This is the same as saying $\lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0$. Both expressions go to infinity, so we may use L'Hôpital's rule to see that the limit is $\lim_{x\to\infty} \frac{(2\ln x)/x}{1}$. Another use of L'Hôpital's rule shows this to equal $\lim_{x\to\infty} \frac{2/x}{1}$, which goes to zero.

8. Compute these integrals. One is a substitution, one is by parts, one is substitution then parts.

(a) $\displaystyle\int x^2 \ln x \, dx$

Solution: Integrate by parts with $u = \ln x$, $du = dx/x$, $dvv = x^2\,dx$ and $v = x^3/3$ to get $\displaystyle\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3}\, dx = \boxed{\frac{x^3}{3} \ln x - \frac{x^3}{9} + C}$.

(b) $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\, dx$

Solution: Substitue $u = 9 + x^2$ and $du = 2x\,dx$ to change the integral into $\displaystyle\int \frac{du/2}{\sqrt{u}}$. This integrates to $\sqrt{u}$, If you compute the limits of integration in terms of u, you see that $0 \le x \le 4$ implies $9 \le u \le 25$, so the definite integral is $\sqrt{25} - \sqrt{9} = 2$. Alternatively, finish the indefinite integral by substituting back in terms of x:

$$\int \frac{du}{2\sqrt{u}} = \sqrt{u} = \sqrt{9+x^2}$$

and evaluate $\sqrt{9+x^2}\Big|_0^4 = \sqrt{25} - \sqrt{9} = \boxed{2}$.

(c) $\displaystyle\int_1^4 e^{\sqrt{x}}\, dx$

Solution: First step: substitute $u = \sqrt{x}, du = dx/(2\sqrt{x}) = du/(2u)$ to turn the integral into $\displaystyle\int_1^2 2ue^u\, du$.

Second step: integrate by parts to get

$$(2ue^u)|_1^2 - \int_1^2 2e^u\, du = (2ue^u)|_1^2 - 2e^u|_1^2 .$$

This evaluates to $(4e^2 - 2e) - (2e^2 - 2e) = \boxed{2e^2}$.

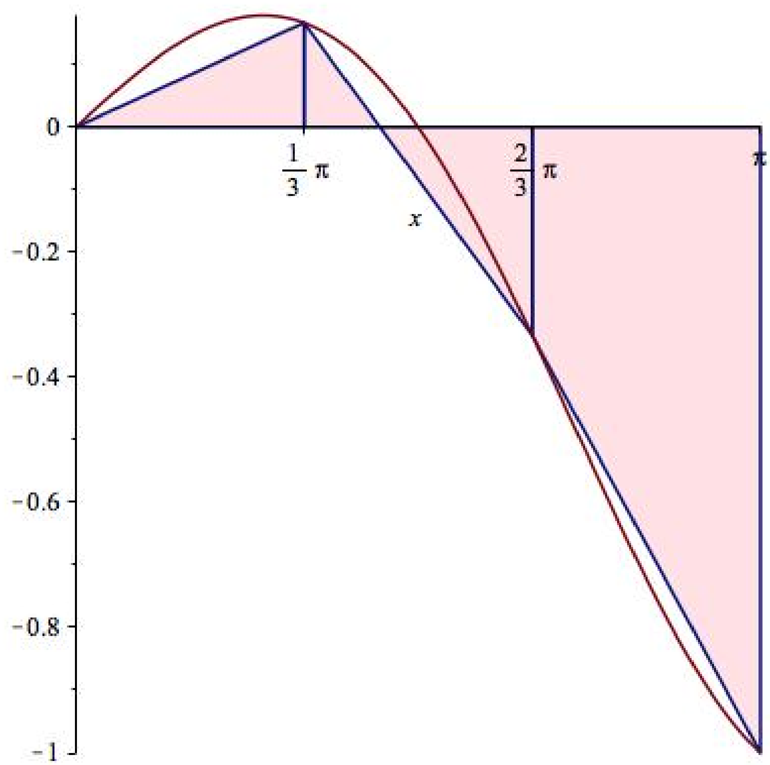
9. Write a trapezoidal approximation for this integral using three trapezoids of equal length. Then evaluate the sum as an exact expression. We do not want a decimal approximation.

$$\int_0^{\pi} \frac{x\cos(x)}{\pi}$$

Solution: The four endpoints of the three equally spaced intervals are $0, \pi/3, 2\pi/3$ and π. The values of $x\cos(x)/\pi$ at these points are respectively $0, 1/6, -1/3$ and -1. The trapezoidal rule calls for summing values at the interior points and half the values at the endpoints then multiplying by the common length of the intervals. Thus,

$$\left(0 + \frac{1}{6} - \frac{1}{3} - \frac{1}{2}\right) \cdot \frac{\pi}{3} = \boxed{\frac{-2}{9}\pi}.$$

If you look at the figure it is kind of funky: the first trapezoid has height zero on the left so it is a triangle. The second trapezoid has a left side that goes up and a right side that goes down, so the diagonal side actally crosses the base, creating a positive area on the left and a negative area on the right. The trapezoidal method above correctly sums these signed areas.



10. John the tax cheat owes the IRS \$100,000. At the end of each year, starting in 2015, a 5% fine is imposed on whatever he owes, after which he is forced to pay \$10,000 (via wage garnishment).

(a) How much does he owe on January 1, 2016?

Solution:

$$\boxed{\$100,000 \cdot 1.05 - \$10,000}$$

(b) How much does he owe on January 1, 2017?

Solution: Multiplying by 1.05 and subtracting \$10,000 gives

$$\boxed{\$100,000 \cdot 1.05^2 - 10,000 \cdot 1.05 - 10,000}.$$

(c) How much does he owe on January 1, 2018?

Solution: Again, multplying by 1.05 and subtracting \$10,000 gives

$$\boxed{\$100,000 \cdot 1.05^3 - 10,000 \cdot (1.05^2 + 1.05 + 1)}.$$

(d) Write an expression in Sigma notation for what Joe owes on January 1, 2025.

Solution:

$$\boxed{\$100,000 \cdot 1.05^{10} - 10,000 \cdot \sum_{n=0}^{9} 1.05^n}$$

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(e) Evaluate this expression analytically by evaluating the summation and simplifying.

Solution: Using the formula for a finite geometric sum gives $\sum_{n=0}^{9} 1.05^n = \frac{1.05^{10} - 1}{1.05 - 1}$. Plugging this into the previous formula gives $\$100,000 \cdot 1.05^{10} - 200,000(1.05^{10} - 1) =$

$$\boxed{\$200,000 - 100,000 \cdot 1.05^{10}}$$

(f) Find a numerical approximation to this expression by using the linearization of $\ln x$ near $x = 1$ and the approximation $e^{1/2} \approx 1.65$. [Hint: if you don't have an $e^{1/2}$ in there somewhere, you have probably made a mistake.]

Solution: Because the derivative of $\ln x$ at $x = 1$ is $1/1 = 1$, the linear approximation of ln near $x = 1$ is $\ln x \approx \ln(1) + (x - 1) \cdot 1 = x - 1$. Therefore $\ln 1.05 \approx 0.05 = 1/20$. This means that

$$1.05^{10} = e^{10 \cdot \ln 1.05} \approx e^{1/2} \approx 1.65\,.$$

Plugging this in gives $\$200,000 - 100,000 \cdot 1.65 = \boxed{\$35{,}000}$.