Name: ____

Midterm Exam II for Math 110, Fall 2015

Solutions

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
10	12	
Total	120	

- You have ninety minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. (a) Find c and p such that $\dfrac{1}{3x+x^2} \sim cx^p$ as $x \to 0$.

Solution: $\boxed{\begin{array}{rcl} c &=& 1/3 \\ p &=& -1 \end{array}}$

As $x \to 0$, $x^2 \ll |3x|$ so x^2 may be ignored and the quantity is asymptotically equivalent to $\dfrac{1}{3x} = \dfrac{1}{3}x^{-1}$. We check this:

$$\begin{aligned} \lim_{x\to 0} \frac{1(3x+x^2)}{(1/3)x^{-1}} &= \lim_{x\to 0} \frac{3x}{3x+x^2} \\ &= \lim_{x\to 0} \frac{1}{1+x/3} \\ &= 1\,. \end{aligned}$$

(b) Write this integral as a limit and say whether or not it is convergent (with justification).

$$\int_0^1 \frac{dx}{3x+x^2}$$

Solution: $\boxed{\text{NO}}$

$$\int_0^1 \frac{dx}{3x+x^2} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{3x+x^2}\,.$$

Because $1/(3x+x^2) \sim (1/3)x^{-1}$ as $x \to 0^+$ and the p-test with $p = -1$ shows this to diverge as $a \to 0^+$, we know that the original integral $\int_0^1 dx/(3x+x^2)$ diverges.

2. Write $\int_0^\infty \frac{dx}{x-5}\,dx$ as a limit or sum of limits, then say whether it converges and why.

Solution: $\boxed{\text{DNE}}$

The integrand is discontinuous at $x = 5$, therefore there are problems at 5 and ∞. We therefore need to cut once arbitrarily at some value K between 5 and $+\infty$ and write

$$\lim_{a \to 5^-} \int_0^a \frac{dx}{x-5}\,dx$$
$$+ \lim_{b \to 5^+} \int_b^K \frac{dx}{x-5}\,dx$$
$$+ \lim_{M \to \infty} \int_K^M \frac{dx}{x-5}\,dx$$

Near 5, we write $1/(x-5) = (x-5)^{-1}$, then use the p-test with $p = -1$ to see that neither of the first two intergals converges. Already this gives us the final answer of "NO". But just in case, the p-test can also be applied to the last integral, giving DNE for that one as well.

3. (a) Compute the value of C that makes $\frac{C}{\sqrt{x}}$ a probability density on $[0, 1]$.

Solution:

$$\int_0^1 \frac{C}{\sqrt{x}}\, dx \;=\; 2Cx^{1/2}\Big|_0^1 \;=\; 2C.$$

Setting this equal to 1 gives $\boxed{C = \frac{1}{2}}$.

(b) Compute the mean of this density.

Solution:

$$\int_0^1 x \frac{dx}{2\sqrt{x}} = \int_0^1 \frac{x^{1/2}}{2}\, dx = \frac{(2/3)x^{3/2}}{2}\Big|_0^1 = \frac{x^{3/2}}{3}\Big|_0^1$$

so the mean is $\boxed{\frac{1}{3}}$.

4. (a) Compute the quadratic MacLaurin polynomial $P_2(x)$ for the function $f(x) = e^{x^2}\sqrt{1+x}$.

Solution: We multiply the MacLaurin series for e^{x^2} by the one for $\sqrt{1+x}$. For e^{x^2}, substituting x^2 for x in the series for e^x gives

$$P_2(x) = 1 + x^2 .$$

For $\sqrt{1+x}$, direct computation (we have done this one before) gives

$$P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 .$$

Multiplying and ignoring terms of degree higher than 2 gives

$$\boxed{P_2(x) = 1 + \frac{1}{2}x + \frac{7}{8}x^2} .$$

(b) What exact value does this give as an approimxation to $e^{1/4}\sqrt{3/2}$?

Solution: Just plug in $x = 1/2$ to get $P_2(1/2) = 1 + \frac{1}{4} + \frac{7}{32} = \boxed{\frac{43}{32}}$.

5. Write a differential equation for this scenario. You do not have to solve the differential equation but you must give the interpretation of all variables and constants, their units, and indicate which is the dependent and the independent variable.

> An old growth forest shrinks due to erosion, human encroachment, disaster and many other factors. The rate per unit time at which the area of this forest decreases is proportional to the square root of the area. However, the Nature Conservancy also adds to it at the rate of one hundred thousand acres per year.

Solution: Let $A(t)$ be the area in acres of old growth forest at time t years. Change in A is due to two sources: growth (shrinkage) at rate $-k\sqrt{A(t)}$ acres per year due to many factors, and growth at rate 100,000 acres per year due to Nature Conservancy investment. Adding these gives

$$\boxed{\frac{dA}{dt} = -kA^{1/2} + 100000}$$

where the constant k is in units of square root of area per time, otherwise known as distance per time, because $A^{1/2}$ has units of distance.

Note: the units of k might have been more obvious had we used an area unit such as square kilometers for A instead of acres. Nevertheless, the square root of acres is a distance unit! Answers describing the units of k as "acres$^{1/2}$/years" or the equivalent received full credit.

6. (a) Does this series converge? Justify your answer.

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

Solution: $\boxed{\text{YES}}$ using the ratio test. Write $a_n = n^2/n!$ and compute

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)^2/n^2}{((n+1)!/n!} \\
&= \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{n+1} \\
&= 0\,.
\end{aligned}$$

The limiting ratio is less than 1 therefore the series converges.

(b) Does this series converge? Justify your answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/4}}$$

Solution: $\boxed{\text{YES}}$.

This is an alternating series with terms that decrease in magnitude and have a limit of zero. Therefore, by the alternating series test, this series does indeed converge.

7. A revenue forecast for a tecnhological innovation forecasts earnings of n^{-2} million dollars in year n.

   (a) Express the total possible earnings of this innovation (assuming you can continue marketing it forever) as an infinite series in Sigma-notation and say whether this total is finite or infinite.

   Solution: The total, in millions of dollars, is $\boxed{\sum_{n=1}^{\infty} n^{-2}}$.

   This is $\boxed{\text{FINITE}}$ by the p-test because $-2 < -1$.

   (b) **You should have answered "finite" in part(a):**

   Approximately how much value remains to be earned after M years? Use an integral approximation and write your answer as cM^p.

   Solution: What remains, in millions of dollars, is $\sum_{n=M+1}^{\infty} n^{-2}$. Approximating by an integral this is roughly $\int_M^{\infty} x^{-2}\,dx = -x^{-1}\big|_M^{\infty} = M^{-1}$. Thus $\boxed{c = 1 \text{ and } p = -1}$.

   Note: One can do more than what is asked, namely get upper and lower bounds for the sum in terms of integrals. The approximation used above is actually and upper bound:

$$\sum_{n=M+1}^{\infty} n^{-2} \le \int_M^{\infty} x^{-2}\,dx = \frac{1}{M}\,.$$

   For a lower bound one could use

$$\sum_{n=M+1}^{\infty} n^{-2} \ge \int_{M+1}^{\infty} x^{-2}\,dx = \frac{1}{M+1}\,.$$

   Both of these bounds are asymptotically M^{-1} as $M \to \infty$.

8. Write an initial value problem for this integral equation.

$$y(t) = 3t + \int_2^t \frac{1}{s + y(s)}\, ds\,.$$

Solution: Differentiating both sides with respect to t,

$$\boxed{y'(t) = 3 + \frac{1}{t + y(t)}}\,.$$

The value of y is known when $t = 2$ and the integral vanishes, leaving $\boxed{y(2) = 6}$.

9. Which of these equations is satisfied by the function $y(t) = (1+t)^{-2}$? Circle any, all or none: 3 points for each correctly circled or not. Answers without justification will be graded all or nothing.

Solution: Computing gives $y' = -2(1+t)^{-3}$. This will be used to test the first three equations.

(i) $y' = \dfrac{-2y}{1+t}$

Solution: $\boxed{\text{YES}}$

$-2(1+t)^{-3} = -2y(1+t)^{-1}$.

(ii) $y' = -2y^{3/2}$

Solution: $\boxed{\text{YES}}$

$$-2(1+t)^{-3} = -2\left[(1+t)^{-2}\right]^{3/2} = -2y^{3/2}\,.$$

(iii) $y' = -2(1+t)y^2$

Solution: $\boxed{\text{YES}}$

$$-2(1+t)^{-3} = -2(1+t)(1+t)^{-4} = -2(1+t)y^2\,.$$

(iv) $y = \displaystyle\int_1^t \frac{-2y}{1+s}\,ds + \frac{1}{4}$

Solution: $\boxed{\text{YES}}$

This integral equation is the same as the differential equation $y' = -2y/(1+t)$ with initial condition $y(1) = 1/4$. We already know from part (1) that $y(t) = (1+t)^{-2}$ solves this differential equation. Also the it satisfies the initial condition $y(1) = 1/4$, therefore it satisfies the IVP that is equivalent to the given integral equation.

Alternatively, plugging in $y(t) = (1+t)^{-2}$ gives the equation

$$(1+t)^{-2} = \frac{1}{4}\int_1^t \frac{-2(1+s)^{-2}}{1+s}\,ds\,. \tag{1}$$

Simplifying the right-hand side and integrating gives $\frac{1}{4} + (1+s)^{-2}\big|_1^t$ which is equal to $(1+t)^{-2}$, verifying equation (1) directly.

10. Use Euler iteration with a step size of 1 to approximate $y(3)$ where $y(t)$ is the solution to the initial value problem

$$y' = x + \sqrt{1+y}\ ; \quad y(1) = 3\,.$$

Solution: Let $F(x,y) = x + \sqrt{1+y}$. The most methodical way to compute the Euler approximation is to fill in this chart. Note that because $\Delta x = 1$, we always have $\Delta y = F(x,y)\Delta x = F(x,y)$, so the last two columns will be the same.

x	y	$F(x,y)$	Δy
1	3	3	3
2	6	$2+\sqrt{7}$	$2+\sqrt{7}$
3	$8+\sqrt{7}$		

Thus Euler's approximation gives $y(3) \approx \boxed{8+\sqrt{7}}$.

Note: the true value of 12.646... is greater than the approximate value of $8+\sqrt{7} \approx 10.6$ because the true curve continues to bend upward. The 4-term Euler approximation (step size = 1/2) does somewhat better, giving the approximation $y(3) \approx 11.6$. It takes about 100 steps before the value goes above 12.6. Apparently convergence is not all that fast.