Name: ____

Midterm Exam II for Math 110, Fall 2015

November 02, 2017

Problem	Points	Score
1	15	
2	16	
3	15	
4	12	
5	12	
6	18	
7	12	
8	12	
9	8	
Total	120	

- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

Signature: ______________________________

Date: ______________________________

1. (a) Find c and p such that $\dfrac{x^{-2}}{2+1/x} \sim cx^p$ as $x \to \infty$.

c = ____________

p = ____________

Show your work or justify your answer.

Solution: Observe that

$$\lim_{x\to\infty} \left(\frac{x^{-2}}{2+1/x}\right) \Big/ \left(\frac{1}{2x^2}\right) = \lim_{x\to\infty} \frac{2}{2+1/x} = 1$$

so we require $c = 1/2$ and $p = -2$.

(b) Write this integral as a limit.

$$\int_1^\infty \frac{x^{-2}}{2+1/x}\,dx$$

Solution: It's simply

$$\lim_{k\to\infty} \int_1^k \frac{x^{-2}}{2+1/x}\,dx.$$

(c) Say (with justification) whether or not the integral in part (b) is convergent.

Solution: Yes, using some combination of the p-test and comparison test.

2. (a) Compute the indefinite integral of x^2e^{-x}.

Solution: Use integration by parts to get

$$\int x^2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + \text{constant}$$

(b) Write $\int_0^\infty x^2e^{-x}\,dx$ as a limit and compute its value using the indefinite integral you computed in part (a). Please briefly justify any assertions about the limit as $M \to \infty$.

Solution: One gets

$$\begin{aligned}\int_0^\infty x^2e^{-x}\,dx &= \lim_{k\to\infty}\int_0^k x^2e^{-x}\,dx \\ &= \lim_{k\to\infty}(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}) - (-2) \\ &= 2.\end{aligned}$$

(c) Compute the value of C that makes Cx^2e^{-x} a probability density on the positive real numbers.

Solution: It's 1/2 by part (b).

(d) Compute the mean of this density.

Solution: Another application of integration by parts tells us that

$$\frac{1}{2}\int_0^\infty x\cdot x^2e^{-x}\,dx = 3.$$

3. (a) Compute the quadratic ($n = 2$) Taylor polynomial $P_2(x)$ for the function $x^{2/3}$ at the value $a = 8$.

Solution: Since $f(x) = x^{2/3}$ has

$$f'(x) = \frac{2x^{-1/3}}{3} \text{ and } f''(x) = -\frac{2x^{-4/3}}{9},$$

one computes

$$\begin{aligned} P_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 4 + \frac{1}{3}(x-8) - \frac{1}{144}(x-8)^2. \end{aligned}$$

(b) What does this give as an approximation to $12^{2/3}$? Write the answer in the box, simplifying fractions when possible, not using any decimal approximations:

$P_2(12) =$ [ ]

Solution: The answer is

$$P_2(12) = 4 + \frac{1}{3}(12-8) - \frac{1}{144}(12-8)^2 = 5 + \frac{4}{9}.$$

(c) State what Taylor's theorem with remainder says about the remainder $R_2 = 12^{2/3} - P_2(12)$. You do not need to compute anything or find bounds, just state the conclusion of the theorem applied to this case.

Solution: One has

$$|R_2| \leq \frac{M(12-8)^3}{3!},$$

where we choose $M \geq \sup\{|f'''(\alpha)| : 8 < \alpha < 12\}$.

4. Write a differential equation for this scenario. You do not have to solve the differential equation but you must give the interpretation of all variables and constants, their units, and indicate which is the dependent and the independent variable.

   The amount of an endowment increases due to investment at a rate proportional to the value, decreased by fees that are taken out at a constant rate.

   Solution: The differential equation should look like

$$\frac{dE}{dt} = kE - F,$$

   where

   - E is the amount of endowment,
   - t is time,
   - k is the constant of proportionality of endowment with respect to rate,
   - F is the fees that are taken out at a constant rate.

5. (a) For which values of x does this series converge?

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$$

Solution: Observe that

$$\lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}/(n+1)!}{2^n x^n/n!} \right| = \lim_{n\to\infty} \left| \frac{2x}{n+1} \right| = 0,$$

so the ratio test tells us that the series converges everywhere on the real line.

(b) Compute the MacLaurin series for e^{2x} and state its radius of convergence.

Solution: From the Maclaurin series of e^x we know that

$$e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$$

and this series converges everywhere on the real line.

6. A revenue forecast for a technological innovation forecasts earnings of $n^{-2/5}$ million dollars in year n.

   (a) Express the total possible earnings of this innovation (assuming you can continue marketing it forever) as an infinite series in Sigma-notation.

   Solution: An observation tells us the series should be

$$\sum_{n=1}^{\infty} n^{-2/5}.$$

   (b) Say whether this total is finite or infinite.

   Solution: It's infinite by the integral test.

   (c) **If you answered "infinite":**

   Approximately how much should you have earned after some large number of years, M? Use an integral approximation and write your answer as cM^p.

   If you answered "finite":

   Approximately how much value remains to be earned after M years? Use an integral approximation and write your answer as cM^p.

   c = ____________

   p = ____________

   Solution: An approximation is

$$\int_1^M x^{-2/5}\,dx = \frac{5}{3}(M^{3/5} - 1),$$

   which is approximately $5M^{3/5}/3$. So one should put $c = 5/3$ and $p = 3/5$.

7. (a) Choose which initial value problem corresponds to this integral equation.

$$y(x) = 5x^2 + \int_3^x \frac{1}{y(t)}\, dt\,.$$

(A) $y' = 5x^2 + 1/y \;\; ; \;\; y(0) = 0$
(B) $y' = 5x^2 + 1/y \;\; ; \;\; y(3) = 30$
(C) $y' = 5x^2 + 1/y \;\; ; \;\; y(3) = 45$
(D) $y' = 5x^2 + 1/y \;\; ; \;\; y(0) = 5x^2$
(E) $y' = 10x + 1/y \;\; ; \;\; y(0) = 0$
(F) $y' = 10x + 1/y \;\; ; \;\; y(3) = 30$
(G) $y' = 10x + 1/y \;\; ; \;\; y(3) = 45$
(H) $y' = 5x^2 + y'/y \;\; ; \;\; y(3) = 0$

Solution: (G).

(b) Choose which integral equation corresponds to this initial value problem.

$$y' = \frac{x}{1+y} \quad ; \quad y(2) = 17\,.$$

(A) $y(x) = \int x/(1+y)\, dx \;\; ; \;\; y(2) = 17$
(B) $y(x) = \int_0^\infty x/(1+y)\, dx$
(C) $y(x) = \int_1^x t/(1+y)\, dt \;\; ; \;\; y(2) = 17$
(D) $y(x) = 17 + \int_2^x t/(1+y)\, dt$
(E) $y(x) = 17 + \int_0^x t/(1+y)\, dt$
(F) $y(x) = \int_0^x t/(1+y)\, dt + \int_0^2 17\, dx$

Solution: (D).

8. Use integrals (hint: you may need to integrate by parts) to find upper and lower bounds on this sum:

$$\sum_{k=1}^{100} \ln(k)\,.$$

Express these bounds as exact quantities involving the natural log. Evaluate one of them approximately by using the log cheatsheet.

Solution: Note that $\ln(1) = 0$, so the sum is secretly from $k = 2$ to $k = 100$. By interpreting the sum above as a Riemann sum, and noting that the graph $f(x) = \ln(x)$ is an increasing function, one gets

$$\int_1^{100} \ln(x)\,dx \leq \sum_{k=1}^{100} \ln(k) \leq \int_2^{101} \ln(x)\,dx$$

where the left inequality is by a right Riemann sum, and the right inequality is by a left Riemann sum. Since, by integration by parts,

$$\int \ln(x)\,dx = x\ln x - x + \text{constant},$$

one gets (using the log cheatsheet)

$$\begin{aligned}\int_1^{100} \ln(x)\,dx &= 100\ln(100) - 100 - (1\ln(1) - 1)\\ &\approx 100\,(2\cdot 2.3) - 100 + 1\\ &= 361.\end{aligned}$$

9. Which of these values comes closest to $y(3)$ if $y(x)$ satisfies the differential equation $y' = (10 + x)\frac{y}{y+1}$ with initial condition $y(1) = 50$? Circle one answer. If you wish to be considered for partial credit, then provide justification.

(A) 0

(B) 72/52

(C) 50

(D) 70

(E) 90

(F) 5,000

Solution: (D). This is because the slope fields at $1 \leq x \leq 3$ and $y \geq 50$ is bounded between 10 and 13, so by a rough linear interpolation

$$70 = y(1) + 10(3 - 1) \leq y(3) \leq y(1) + 13(3 - 1) = 76.$$

So the value 70 comes closest to $y(3)$.

TABLE 8.1 Basic integration formulas

1. $\int k\, dx = kx + C \quad$ (any number k)
2. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln |x| + C$
4. $\int e^x\, dx = e^x + C$
5. $\int a^x\, dx = \dfrac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$
6. $\int \sin x\, dx = -\cos x + C$
7. $\int \cos x\, dx = \sin x + C$
8. $\int \sec^2 x\, dx = \tan x + C$
9. $\int \csc^2 x\, dx = -\cot x + C$
10. $\int \sec x \tan x\, dx = \sec x + C$
11. $\int \csc x \cot x\, dx = -\csc x + C$
12. $\int \tan x\, dx = \ln |\sec x| + C$
13. $\int \cot x\, dx = \ln |\sin x| + C$
14. $\int \sec x\, dx = \ln |\sec x + \tan x| + C$
15. $\int \csc x\, dx = -\ln |\csc x + \cot x| + C$
16. $\int \sinh x\, dx = \cosh x + C$
17. $\int \cosh x\, dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a} \sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \quad (a >$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \quad (x >$

Logarithm Cheat Sheet

These values are accurate to within 1%:

$$\begin{aligned} e &\approx 2.7 \\ \ln(2) &\approx 0.7 \\ \ln(10) &\approx 2.3 \\ \log_{10}(2) &\approx 0.3 \\ \log_{10}(3) &\approx 0.48 \end{aligned}$$

Some other useful quantities to with 1%:

$$\begin{aligned} \pi &\approx \frac{22}{7} \\ \sqrt{10} &\approx \pi \\ \sqrt{2} &\approx 1.4 \\ \sqrt{1/2} &\approx 0.7 \end{aligned}$$

(ok so technically $\sqrt{2}$ is about 1.005% greater than 1.4 and 0.7 is about 1.005% less than $\sqrt{1/2}$)