Name: ____

Midterm Exam III for Math 110, Fall 2015

Solutions

Problem	Points	Score
1	16	
2	12	
3	16	
4	16	
5	12	
6	12	
7	16	
8	12	
9	8	
Total	120	

- You have ninety minutes for this exam.
- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

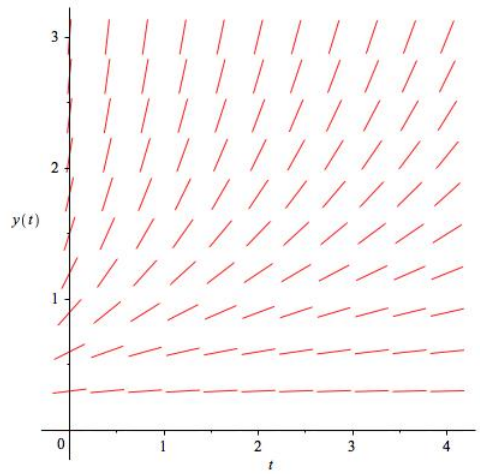
Signature: ______________________________

Date: ______________________________

1. (a) Sketch the slope field in the first quadrant for the differential equation

$$\frac{dy}{dt} = \frac{y^2}{1+t}\,.$$

Solution:



(b) Find the general solution.

Solution: This is a separable equation. Write it as $\dfrac{dy}{y^2} = (1+t)dt$. Then solve:

$$\begin{aligned} -\frac{1}{y} &= \ln(1+t) + C \\ \frac{1}{y} &= C - \ln(1+t) \quad \text{(different C)} \\ y &= \boxed{\frac{1}{C - \ln(1+t)}} \end{aligned}$$

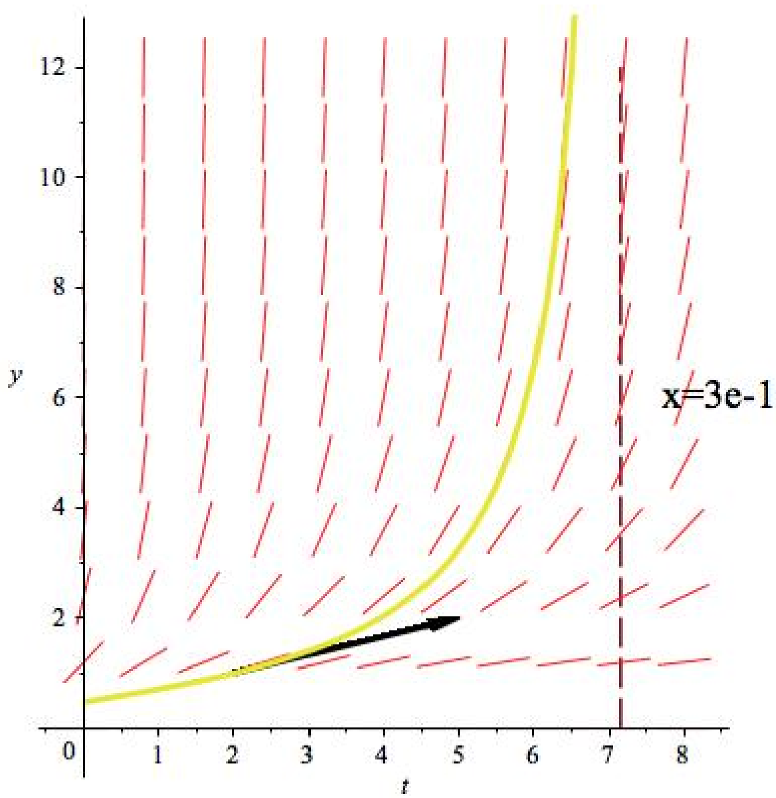
(c) Find the solution with $y(2) = 1$.

Solution: Setting $t = 2$ and $y = 1$ gives $1 = 1/(C - \ln(1+2))$ therefore $\boxed{C = 1 + \ln 3}$.

(d) Sktech this solution, indicating the slope at $t = 2$ and any asymptotes.

Solution: The slope at $(1, 2)$ is $(1)^2/(1 + 2) = \boxed{\frac{1}{3}}$.

There is a vertical asymptote where the denominator vanishes, which occurs when $\ln(1 + t) = 1 + \ln 3$, that is when $\boxed{t = 3e - 1}$.



2. Compute

$$\int_R \frac{y}{1+x^2}\, dA$$

where R is the rectangle $-2 \le y \le 3, 1 \le x \le \sqrt{3}$.

Solution: The easiest method is via the product formula:

$$\begin{aligned}
\int_R \frac{y}{1+x^2}\, dA &= \left(\int_{-2}^{3} y\, dy\right)\left(\int_1^{\sqrt{3}} \frac{1}{1+x^2}\right) \\
&= \left(\frac{y^2}{2}\bigg|_{-2}^{3}\right) \times \left(\arctan x|_1^{\sqrt{3}}\right) \\
&\qquad \text{(see the integral table included with the exam)} \\
&= \frac{5}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \boxed{\frac{5\pi}{24}}.
\end{aligned}$$

3. A retirement fund grows at 3% per year (continuous growth rate) and also by the addition of \$90,000 per year (also added continuously). It starts today with no money in it.

   (a) Write an initial value problem for this. As usual, state the interpretation and units of all variables and constants.

   Solution: Let t be the time in years, starting with today $= 0$, and let $M(t)$ be the amount in the fund after time t in thousands of dollars. Then

$$\frac{dM}{dt} = 0.03\text{years}^{-1} \cdot M(t) + 90$$

   with the initial condition $M(0) = 0$.

   (b) Solve the IVP.

   Solution: This is a linear first order differential equation $M' + P(t)M = Q(t)$ where $P = -0.03$ and $Q = 90$ are both constants that don't depend on t. using the integrating factor $e^{\int P(t)\,dt} = e^{-0.03t}$ we get

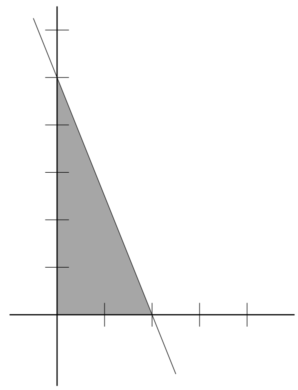
$$e^{-0.03t}M' - 0.03e^{-0.03t}M = 90e^{-0.03t}.$$

   Integrating, we know the left-hand side becomes $e^{-0.03t}M$, hence

$$e^{-0.03t}M = -3000e^{-0.03t} + C$$

   which yields the solution $M(t) = Ce^{0.03t} - 3000$. Applying the initial condition $C = 3000$ so $\boxed{M(t) = 3000\left(e^{0.03t} - 1\right)}$.

4. A dart lands uniformly somewhere in the triangle bounded by the x-axis, the y-axis and the line $5x + 2y = 10$.



(a) Compute the probability density of the location of the dart.

Solution: The density for the uniform distribution on a region is the constant $1/A$ on the region, where A is the area. The area of a triangle with base 2 and height 5 is 5, so the density is $\boxed{f(x, y) = \frac{1}{5}}$ on the region $x \geq 0, y \geq 0, 5x + 2y \leq 10$.

(b) Compute the mean of the y-coordinate of the dart.

Solution:

$$
\begin{aligned}
\int_R \frac{1}{5} y \, dA &= \frac{1}{5} \int_0^2 \int_0^{5-(5/2)x} y \, dy \, dx \\
&= \frac{1}{5} \int_0^2 \left. \frac{y^2}{2} \right|_0^{5-(5/2)x} dx \\
&= \frac{1}{10} \int_0^2 (5 - (5/2)x)^2 \, dx \\
&= \frac{1}{10} \frac{-2}{5} \frac{1}{3} (5 - (5/2)x)^3 \Big|_0^2 \\
&= \frac{2}{150} (5^3 - 0^3) \\
&= \boxed{\frac{5}{3}}.
\end{aligned}
$$

5. Use the increment theorem with $g(x, y) = \ln(x - \sqrt{y})$ to give a decimal approximation to $\ln(4.1 - \sqrt{8.8})$.

Solution: We need to use the theorem at a nearby point at which we can evaluate the function; in this case the point is $(4, 9)$ and $g(4, 9) = \ln 1 = 0$. We compute

$$\frac{\partial g}{\partial x} = \frac{1}{x - \sqrt{y}} \quad ; \quad \frac{\partial g}{\partial x}(4, 9) = 1$$

and

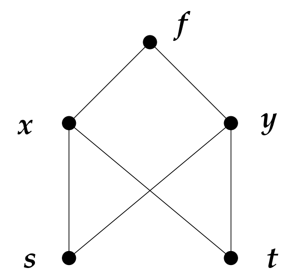
$$\frac{\partial g}{\partial y} = \frac{-1/(2\sqrt{y})}{x - \sqrt{y}} \quad ; \quad \frac{\partial g}{\partial y}(4, 9) = -\frac{1}{6} .$$

The increment theorem then yields

$$g(4.1, 8.8) \approx 0 + (0.1) \cdot 1 - \frac{1}{6} \cdot (-0.2) \approx .13333\ldots .$$

Therefore, $\boxed{\ln(4.1 - \sqrt{8.8}) \approx 0.13}$.

6. The following branch diagram refers to a function f of two variables x and y, each of which is a function of two other variables s and t.



(i) When evaluating the expression $\partial f/\partial s$, which of the variables s, t, x, y vary independently, which stay fixed, and which vary dependently?

Solution: The independent variable is s. The variable t (another independent variable) is held constant. The variables x and y (intermediate variables) vary when you vary s.

(ii) Evaluate $\partial f/\partial s$ at the point $(s, t) = (1, 4)$ if the functions in question are given by these equations:

$$\begin{aligned} f(x, y) &= \frac{x^2}{1+y} \\ x(s, t) &= e^{s+t-1} \\ y(s, t) &= s + \sqrt{t} \end{aligned}$$

Solution: We compute

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x}{1+y} \\ \frac{\partial f}{\partial y} &= \frac{-x^2}{(1+y)^2} \\ \frac{\partial x}{\partial s} &= e^{s+t-1} \\ \frac{\partial y}{\partial s} &= 1 \end{aligned}$$

At $(s, t) = (1, 4)$ we know that $x = e^4$ and $y = 3$. Plugging into the equations above, $f_x = e^4/2, f_y = -e^8/16, x_s = e^4, y_s = 1$. Therefore

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = e^8/2 - e^8/16 = \boxed{\frac{7e^8}{16}}.$$

7. Suppose that the quantity W of widgets that can be manufactured per unit time is a function of billions of dollars D invested in equipment and the number N of employees, via the formula

$$W = 10,000\,(D-3)\frac{N}{1500+N}\,.$$

The management is thinking of saving on annual labor cost (reducing N) by a one-time capital investment (increasing D). They need to keep production (W) constant.

How much more money would have to be invested in equipment per employee laid off if the present investment is 4.05 billion dollars and there are presently 2000 employees?

Solution: The marginal rate of substitution of billions of dollars per employee is $-dD/dN$, which by implicit differentiation is equal to W_N/W_D. We compute:

$$\frac{\partial W}{\partial D} = 10000\frac{N}{1500+N}$$

and

$$\frac{\partial W}{\partial N} = 10000(D-3)\frac{1500}{(1500+N)^2}$$

therefore the marginal rate of substitution is

$$\frac{1}{N}\frac{1500(D-3)}{1500+N}\,.$$

Evaluating at $D = 81/20$ and $N = 2000$ gives $(21/20)(4/7) =$

$$\frac{1}{2000}\frac{1500}{3500}\frac{21}{20} = \frac{1}{2000}\frac{9}{20} = \boxed{\frac{9}{40000}\text{ billion}}$$

or $\boxed{\$225{,}000}$ more capital investment dollars per fewer employee.

8. A function f has the following values. Estimate the value of $\frac{\partial f}{\partial x}(7,3)$.

x	y	$f(x,y)$
7	3	10
7	5	18
6	3	12
3	7	20
70	30	100

Solution: We need the values at $(7,3)$ and a point for which the x-coordinate is near 7 and the y-coordinate is still 3. Using the first and third line, $f(7,3) = 10$ and $f(6,3) = 12$, therefore $\Delta f = 2$ when $\Delta x = -1$ yielding an estimate of $\boxed{\frac{\partial f}{\partial x}(7,3) = -2}$.

9. Let R be the unit square $0 \le x \le 1, 0 \le y \le 1$. Approximate

$$\int_R \sqrt{x^2 + y^2}\, dA$$

by using a Riemann sum in which the unit square is divided into four subsquares in a 2×2 grid and the function is evaluated at the center of each subsquare. Please leave the answer as an analytic expression rather than a decimal approximation.

Solution: We use the midpoints $(1/4, 1/4), (1/4, 3/4), (3/4, 1/4), (3/4, 3/4)$ for the four subsquares. Evaluating there gives $\sqrt{x^2 + y^2}$ the values respectively $\frac{\sqrt{2}}{4}, \frac{\sqrt{10}}{4}, \frac{\sqrt{10}}{4}$ and $\frac{\sqrt{18}}{4}$. Multiply each by $1/4$ (the area of the subsquare and sum, yielding $\boxed{\frac{\sqrt{18} + \sqrt{2} + 2\sqrt{10}}{16}}$.

FYI the decimal value of this expression is roughly 0.7488 while the exact value of the integral is

$$1/6 \ln\left(1 + \sqrt{2}\right) + 1/3\sqrt{2} - 1/12 \operatorname{arctanh}\left(1/2\sqrt{2}\right) - 1/4 \ln\left(\sqrt{2} - 1\right)$$

which comes out to roughly 0.7519.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad (\text{any number } k)$

2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

3. $\int \frac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \tan x\,dx = \ln|\sec x| + C$

13. $\int \cot x\,dx = \ln|\sin x| + C$

14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

16. $\int \sinh x\,dx = \cosh x + C$

17. $\int \cosh x\,dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a >$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x >$