Name: ____

Midterm Exam III for Math 110, Fall 2017

Solutions

Problem	Points	Score
1	18	
2	16	
3	12	
4	16	
5	14	
6	16	
7	12	
8	16	
Total	120	

- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- **CLEARLY** indicate final answers. Use words (doesn't have to be that many words) to connect mathematical formulas and equations.
- **NO** books, notes, laptops, cell phones, calculators, or any other electronic devices may be used during the exam. One 8.5×11 cheatsheet, handwritten is allowed; it may be double-sided.
- No form of cheating will be tolerated. You are expected to uphold the Code of Academic Integrity.

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this midterm examination.

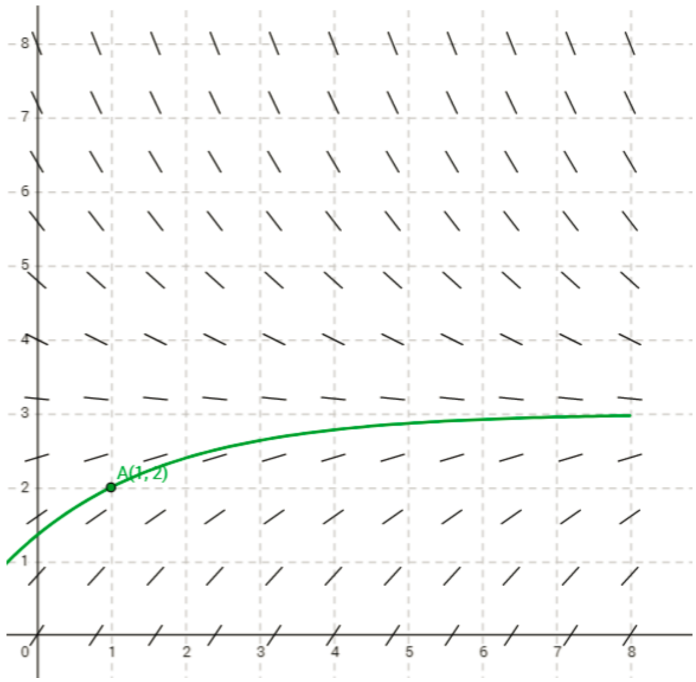
Signature: ________________________________

Date: ________________________________

1. (a) Sketch the slope field in the first quadrant for the differential equation

$$\frac{dy}{dt} = -\frac{y-3}{2}.$$

Solution:



(b) On the same picture, sketch the particular solution with $y(1) = 2$.

Solution: See part (a).

(c) Use Euler iteration with step size 1/2 to compute an approximation to $y(2)$ for this particular solution.

Solution: Use $y(1) = 2, y^{'}(1) = -\frac{y(1)-3}{2} = \frac{1}{2}$, we obtain:

$$y(3/2) \approx y(1) + y^{'}(1) \times \frac{1}{2} = \frac{9}{4};$$

Therefore, $y^{'}(3/2) = -\frac{y(3/2)-3}{2} \approx \frac{3}{8}$. Finally,

$$y(2) \approx y(3/2) + y^{'}(3/2) \times \frac{1}{2} \approx \frac{39}{16} = 2.4375.$$

(d) Compute the general solution.

Solution:

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{y-3}{2} \\
\frac{dy}{y-3} &= -\frac{1}{2}dt \\
\int \frac{dy}{y-3} &= -\int \frac{1}{2}dt + C \\
\ln|y-3| &= -\frac{1}{2}t + C \\
y - 3 &= C^{'}e^{-t/2} \\
y &= 3 + C^{'}e^{-t/2}
\end{aligned}$$

(e) Compute the particular solution with $y(2) = 1$.

Solution: Plug in $(t, y) = (2, 1)$ in the general solution, we have:

$$1 = 3 + C^{'}e^{-1} \Rightarrow C^{'} = -2e.$$

Therefore, the particular solution is

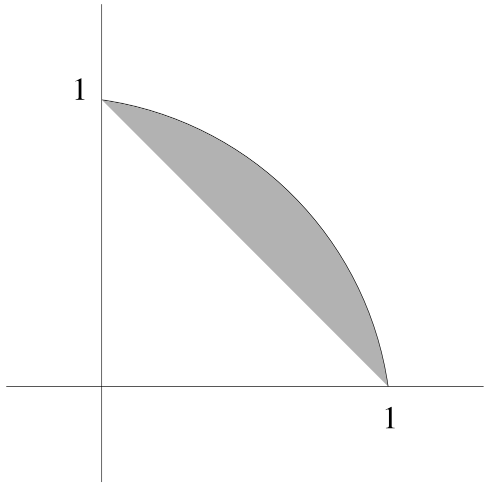
$$y = 3 - 2e^{1-t/2}.$$

(f) For this solution, find $\lim_{t\to\infty} y(t)$ or prove it does not exist.

Solution:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 3 - 2e^{1-t/2} = 3.$$

2. Let R be the region inside the unit circle in the first quadrant and above the line $x + y = 1$, as shown in the figure.



(a) Describe R in the form $\{(x, y) : \cdots \}$ using vertical strips.

Solution:

$$R = \{(x, y) : 0 \le x \le 1, 1 - x \le y \le \sqrt{1 - x^2}\}$$

(b) Suppose gold is buried in the region R and the density of gold in ounces per square meter at the point (x, y) is $\sqrt{1 - x^2} + (1 - x)$. What is the total amount of gold in the region?

Solution: The total amount of gold in the region is

$$\begin{aligned}
&\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} \sqrt{1 - x^2} + (1 - x) dy dx \\
=&\int_0^1 (\sqrt{1 - x^2} + (1 - x))(\sqrt{1 - x^2} - (1 - x)) dx \\
=&\int_0^1 2x - 2x^2 dx \\
=&\frac{1}{3}.
\end{aligned}$$

3. Circle the number of the correct solution to the initial value problem

$$y' = y\frac{\sin x}{x^2}\ ;\ \ y(2) = 3\,.$$

(i) $y = 3e^{2-x}$

(ii) $y = \ln\left(e^3 + \int_2^x \frac{\sin t}{t^2}\,dt\right)$

(iii) $y = e^{3+\int_2^x \frac{\sin t}{t^2}\,dt}$

(iv) $y = 3\int \frac{\sin x}{x^2}\,dx$

(v) $y = e^{\int \frac{\sin x}{x^2}\,dx+C}$

(vi) $y = 3 + e^{\int_2^x \frac{\sin t}{t^2}\,dt}$

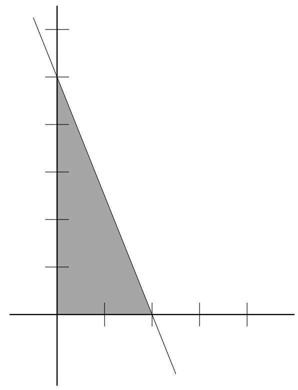
(vii) $y = 3e^{\int_2^x \frac{\sin t}{t^2}\,dt}$

Solution:

$$\begin{aligned} y' &= y\frac{\sin x}{x^2} \\ \frac{dy}{y} &= \frac{\sin x}{x^2}dx \\ \int \frac{dy}{y} &= \int \frac{\sin x}{x^2}dx + C \\ \ln|y| &= \int \frac{\sin x}{x^2}dx + C \\ y &= C'e^{\int \frac{\sin x}{x^2}dx} \end{aligned}$$

Answers (iii), (v) and (vii) are all of the correct form but (iii) and (v) have the wrong values of C to satisfy $y(2) = 3$, therefore the correct answer is (vii).

4. A dart lands uniformly somewhere in the triangle bounded by the x-axis, the y-axis and the line $5x + 2y = 10$.



(a) Compute the probability density of the location of the dart.

Solution: Since the dart lands uniformly inside the shadow area, the density is just $1/S_{shadow} = 1/5$ anywhere insider the shadow area, and zero elsewhere.

(b) Compute the mean of the y-coordinate of the dart.

Solution: The mean of the y-coordinate is calculated as follows:

$$\begin{aligned}
&\int_0^5 \int_0^{2-2y/5} \frac{1}{5} y dx dy \\
=&\int_0^5 \frac{2}{5}y - \frac{2}{25}y^2 dy \\
=&5 - \frac{10}{3} \\
=&\frac{5}{3}.
\end{aligned}$$

5. Use the increment theorem to give a decimal approximation to $\sqrt{8.9 + \ln(0.8)}$.

Please say what you are choosing for the function $f(x, y)$ and for $x_0, y_0, \Delta x, \Delta y$ and show your computation of the partial derivatives of f at a general point before plugging in specific values.

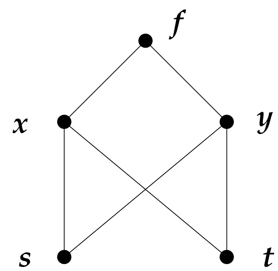
Solution:

$$\begin{aligned} f(x, y) &= \sqrt{x + \ln y}; \\ x_0 &= 9, y_0 = 1; \\ \Delta x &= -0.1, \Delta y = -0.2; \\ \frac{\partial f}{\partial x} &= \frac{1}{2\sqrt{x + \ln y}}; \\ \frac{\partial f}{\partial y} &= \frac{1}{2y\sqrt{x + \ln y}}. \end{aligned}$$

Therefore, we obtain an approximation using the increment theorem:

$$\begin{aligned} f(8.9, 0.8) &\approx f(9, 1) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ &= 3 + \frac{1}{6} \cdot (-0.1) + \frac{1}{6} \cdot (-0.2) \\ &= 2.95. \end{aligned}$$

6. The following branch diagram refers to a function f of two variables x and y, each of which is a function of two other variables s and t.



(a) Write an expression using partial derivatives of f, x and y that computes the rate of change of f per change in s when t remains constant.

Solution:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

(b) Evaluate this at the point $(s,t) = (1,4)$ if the functions in question are given by these equations:

$$\begin{aligned} f(x,y) &= \frac{x^2}{1+y} \\ x(s,t) &= e^{s+t-1} \\ y(s,t) &= s+\sqrt{t} \end{aligned}$$

Solution: First, at the point $(s,t) = (1,4)$, we have $(x,y) = (e^4, 3)$ according to the formulas. Thus,

$$\begin{aligned} \frac{\partial f}{\partial s}(1,4) &= \frac{\partial f}{\partial x}(e^4,3) \cdot \frac{\partial x}{\partial s}(1,4) + \frac{\partial f}{\partial y}(e^4,3) \cdot \frac{\partial y}{\partial s}(1,4) \\ &= \frac{2e^4}{1+3} \cdot e^{1+4-1} - \frac{e^8}{(1+3)^2} \cdot 1 \\ &= \frac{7}{16}e^8. \end{aligned}$$

7. Find the two points where the curve $x^2 - xy + y^3 = 1$ crosses the x-axis. Use implicit differentiation to find the slopes of the tangents at these two points. Do the tangents intersect, or do they coincide, or are they parallel without coinciding?

Solution: Let $y = 0$, we have $x^2 - 1 = 0 \Rightarrow x = \pm 1$. Thus, the two points crossing the x-axis are $(1, 0)$ and $(-1, 0)$.
Next, taking the derivatives in terms of x on both sides of the curve at these two points, we have:

$$2x - y - xy' + 3y^2y' = 0,$$

$$\Rightarrow 2 - y' = 0 \quad \text{at } (1, 0);$$

$$\Rightarrow -2 + y' = 0 \quad \text{at } (-1, 0).$$

Hence both slopes equal to 2, plus these two lines cross two different points, i.e. $(1, 0)$ and $(-1, 0)$ respectively, we conclude that they parallel without coinciding.

8. The sales S of a lite beer, in millions, are predicted by the formula

$$S = \frac{30}{C(101 - T)}$$

where T is the taste score on a scale from 1 to 100 given by *Beer Enthusiast* magazine and C is the number of calories in a 12 oz. bottle.

(a) Compute $\frac{\partial S}{\partial C}$ and $\frac{\partial S}{\partial T}$ at the point $C = 100, T = 96$.

Solution:

$$\frac{\partial S}{\partial C} = -\frac{30}{C^2(101 - T)};$$

$$\frac{\partial S}{\partial T} = \frac{30}{C(101 - T)^2}.$$

Plug in $(C, T) = (100, 96)$, we obtain:

$$\frac{\partial S}{\partial C}(100, 96) = -\frac{6}{100^2} = -\frac{3}{5000};$$

$$\frac{\partial S}{\partial T}(100, 96) = \frac{6}{500} = \frac{3}{250}.$$

(b) If the brewery wants to maintain the same sales, but the taste index slips a little from 96, how must the caloric content change per point of change in taste index?

Solution: When S is held constant and C is viewed as a function of T, the marginal rate of substitution is $-\frac{dC}{dT} = \frac{\partial S/\partial T}{\partial S/\partial C}$. Using the previous computation, this is $(3/250)/(3/5000) = 20$ calories per point in the taste index.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$ $(a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C$ $(a$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C$ $(x$