Spring 2018 Preliminary Exam – Problems and Solutions

1. For each of the following series, either prove that it converges or prove that it diverges.

   a) $1 + 1/2 - 1/3 + 1/4 + 1/5 - 1/6 + 1/7 + 1/8 - 1/9 + 1/10 + 1/11 - 1/12 + 1/13 + 1/14 - 1/15 + \cdots$

   b) $1 + 1/2 + 1/3 - 1 + 1/4 + 1/5 + 1/6 - 1/2 + 1/7 + 1/8 + 1/9 - 1/3 + 1/10 + 1/11 + 1/12 - 1/4 + \cdots$

   Solution:

   (a) It diverges. The sum of the first $3n$ terms is greater than $1 + 1/4 + 1/7 + \cdots + 1/(3n-2) > 1/3 + 1/6 + 1/9 + \cdots + 1/3n$, which becomes arbitrarily large (one-third of the harmonic series).

   (b) It converges. Let S_n be the sum of the first n terms. Then

   $$S_{4k} = 1 + 1/2 + \ldots + 1/(3k) - (1 + 1/2 + \ldots 1/k) = 1/(k+1) + 1/(k+2) + \ldots + 1/(3k).$$

   Thus
   $$\int_{k+1}^{3k+1} 1/x\,dx < S_{4k} < \int_{k}^{3k} 1/x\,dx = \ln(3k) - \ln(k) = \ln(3),$$
   where $\int_{k+1}^{3k+1} 1/x\,dx = \ln(3k+1) - \ln(k+1) = \ln(3 - \frac{2}{k+1})$; and so the sequence S_{4k} converges to $\ln(3)$. For $i = 1, 2, 3$, $|S_{4k+i} - S_{4k}| < 3/(3k+1)$, which approaches 0 as $k \to \infty$; so the sequence S_n also converges to $\ln(3)$. Thus the sum converges (to $\ln(3)$).

2. Consider the matrix
   $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & a & a & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix},$$
   where $a \in \mathbb{R}$.

   a) Determine all values of $a \in \mathbb{R}$ for which the matrix A is invertible.

   b) For each such a, find the determinant of A.

   Solution:

   (a) The matrix A is invertible iff its row rank is 4. Performing elementary row operations yields
   $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & a & a & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2-a \\ 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2-a \end{bmatrix},$$
   which is invertible iff $a \neq 2$.

   (b) The above row operations multiplied the determinant by -1 (because of interchanging rows), and so $\det(A) = -1 \cdot 1 \cdot 2 \cdot (2-a) = 2a - 4$.

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, and that

$$|f(x) - f(y)| \geq |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

a) Prove that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that the compositions $f \circ g$ and $g \circ f$ are both equal to the identity map.

b) Prove that the above function g is continuously differentiable.

Solution:

(a) First, the map f is one-to-one (injective). Indeed, if $f(x) = f(y)$, then $|x - y| \leq |f(x) - f(y)| = 0$, i.e. $x = y$. To prove that g exists, it suffices to prove that f is also onto (surjective), since then f is bijective and thus invertible.

The hypothesis on f implies that $|f'(x)| \geq 1$ for all x, since $f'(x) = \lim_{y \to x}(f(y) - f(x))/(y - x)$. In particular, $f'(x)$ is never equal to 0. Since f' is continuous, the Intermediate Value Theorem implies that either $f'(x) \geq 1$ for all x or $f'(x) \leq -1$ for all x. Possibly after replacing f by $-f$, we may assume the former. After replacing f by $f - f(0)$, we may assume that $f(0) = 0$. So for every $N > 0$, by the Mean Value Theorem there exists c with $f(N)/N = (f(N) - f(0))/(N - 0) = f'(c) \geq 1$; i.e., $f(N) \geq N$. Thus f takes on arbitrarily large positive values; and by the intermediate value theorem, it takes on all positive values, since $f(0) = 0$. Similarly, it takes on all negative values. Hence it is surjective, and thus invertible.

(b) This is immediate from the Inverse Function Theorem, since f' is never equal to 0.

4. Let G be a group of order 66.

a) Find an integer n with $1 < n < 66$ such that G must have a normal subgroup N of index equal to n. Justify your assertion.

b) For this value of n, prove that every $g \in G$ has the property that $g^n \in N$.

Solution:

(a) $n = 6$ works. Namely, by the Sylow theorems, the number of Sylow 11-subgroups is congruent to 1 modulo 11 and divides $66/11 = 6$, and so equals 1. The unique Sylow 11-subgroup is thus normal, of order 11 and index 6.

(b) If $g \in G$, let $\bar{g}$ be its image in G/N. Since N has index equal to 6, G/N has order equal to 6; and so $\bar{g}^6$ is equal to the identity. That is, $g^6 \in N$.

5. Let $a, b \in \mathbb{R}$ and consider the differential equation $f''(x) + af'(x) + bf(x) = 0$. For which values of a, b does there exist a non-zero solution $f : \mathbb{R} \to \mathbb{R}$ to this equation such that f is bounded on $[0, \infty)$? For each such a, b, find such a solution.

Solution:

Answer: Either $a^2 - 4b \geq 0$ and $-a \leq \sqrt{a^2 - 4b}$; or else $a^2 - 4b < 0$ and $a \geq 0$.

If the associated quadratic equation $z^2 + az + b$ has distinct real roots $r_1 < r_2$ (i.e., if $a^2 - 4b > 0$), then the solutions are of the form $C_1 e^{r_1 x} + C_2 e^{r_2 x}$, with $C_1, C_2 \in \mathbb{R}$. For $r \in \mathbb{R}$, the function e^{rx} is bounded on $[0, \infty)$ iff $r \leq 0$. Since the solution is assumed non-zero, either C_1 or C_2 is non-zero, and so the condition is that at the smaller root r_1 is ≤ 0; i.e., $-a \leq \sqrt{a^2 - 4b}$, in which case $e^{r_1 x}$ is a bounded solution.

If $z^2 + az + b$ has a double real root r (i.e., if $a^2 = 4b$), then the solutions are of the form $C_1 e^{rx} + C_2 x e^{rx}$. The function $x e^{rx}$ also is bounded on $[0, \infty)$ iff $r \leq 0$. So in this case the equation has a bounded solution on $[0, \infty)$ iff $r \leq 0$; i.e., $a \geq 0$, in which case e^{rx} is a solution. So for this case and the previous case, the condition is that $a^2 - 4b \geq 0$ and $-a \leq \sqrt{a^2 - 4b}$.

If $z^2 + az + b$ has non-real complex conjugate roots $r \pm is$ (i.e., if $a^2 - 4b < 0$), then the solutions are of the form $C_1 e^{rx} \cos(sx) + C_2 e^{rx} \sin(sx)$. Since e^{rx} is bounded on $[0, \infty)$ iff $r \leq 0$, and since cos and sin are bounded, there is a bounded solution on $[0, \infty)$ iff $r \leq 0$; i.e., $a \geq 0$, in which case $e^{rx} \cos(sx)$ is a bounded solution.

6. For each of the following, either give an example or prove that no such example exists.

   a) A closed subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

   b) An open subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

   c) A connected subset $S \subset \mathbb{R}$ that contains $\mathbb{Q}$, such that $S \neq \mathbb{R}$.

   Solution:

   (a) No such example exists, because $\mathbb{Q}$ is dense in $\mathbb{R}$, and so any closed set that contains $\mathbb{Q}$ must be all of $\mathbb{R}$.

   (b) There are many such examples, e.g., $\{x \in \mathbb{R} \,|\, x \neq \sqrt{2}\}$.

   (c) No such example exists. Since a connected subset of $\mathbb{R}$ (which is a metric space) is path connected, if $a, b \in S$ with $a < b$, then $[a, b] \subseteq S$. Since every real number lies between two rational numbers, all real numbers lie in S.

7. For each continuous function $f(x, y)$ on the x, y-plane, and each path C from $(0, 1)$ to $(\pi, 1)$, consider the contour integral

$$\int_C y \sin^2(x)\, dx + f(x, y)\, dy.$$

   a) Find a choice of the function $f(x, y)$ such that the value of the above integral is independent of the choice of the path C from $(0, 1)$ to $(\pi, 1)$.

   b) For your choice of f, evaluate the above integral for any choice of path C as above.

   Solution:

   (a) By Green's Theorem, the integral has the required property if $f(x, y)$ is defined on $\mathbb{R}^2$ and $\frac{d}{dx} f(x, y) = \frac{d}{dy} y \sin^2(x) = \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$; i.e., if $f(x, y) = \frac{1}{2}(x - \frac{1}{2}\sin(2x)) + g(y)$, where g is a function of y. So we may take $f(x, y) = \frac{1}{2}(x - \frac{1}{2}\sin(2x))$.

(b) We may take the path given by $x = t$, $y = 1$ for $0 \le t \le \pi$. Thus $dx = \frac{dx}{dt}dt = dt$ and $dy = \frac{dy}{dt}dt = 0$. Then the second term drops out and we are left with

$$\int_0^\pi \sin^2(t)dt = \int_0^\pi \frac{1}{2}(1-\cos(2t))dt = \frac{1}{2}\left[(t - \frac{1}{2}\sin(2t))\right]_0^\pi = \frac{1}{2}(\pi - 0) = \pi/2.$$

(If in part (a), a function $f(x,y) = \frac{1}{2}(x - \frac{1}{2}\sin(2x)) + g(y)$ is chosen, then the integral of the first term over the above path is unchanged and that of the second term remains 0, so the answer is still $\pi/2$.)

8. Let q be a power of a prime number, and let $\mathbb{F}_q$ be the field of q elements. Let V be an n-dimensional vector space over $\mathbb{F}_q$. For each positive integer k, let S_k be the set of ordered k-tuples $(v_1, \dots, v_k)$ of linearly independent vectors in V.

   a) Show that the number of elements in S_k is $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$. What does this say if $k > n$?

   b) Using part (a), determine the number of invertible $n \times n$ matrices over $\mathbb{F}_q$.

   Solution:

   (a) There are q^n elements of V. The only constraint on v_1 is that it is non-zero; so there are $q^n - 1$ choices for v_1. Once v_1 is chosen, there are exactly q vectors that are multiples of v_1, and so there are $q^n - q$ allowable choices for v_2. For each choice of v_1, v_2, there are exactly q^2 vectors that are linearly dependent on $\{v_1, v_2\}$, leaving $q^n - q^2$ linearly independent choices of v_3; and so on. So the number of elements in S_k, i.e. the number of choices for $(v_1, \dots, v_k)$, is $(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})$. If $k > n$, this number is 0, as expected, since any set of more than n vectors in an n-dimensional vector space is linearly dependent.

   (b) By considering the columns (or rows), these matrices are in bijection with the elements of S_n. So the number of these matrices is $(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$.

9. Let $I \subset \mathbb{R}$ be an open interval, and let $f : I \to \mathbb{R}$ be a twice differentiable function. Suppose that $a, b, c \in I$ are distinct, and that the three points

$$(a, f(a)), (b, f(b)), (c, f(c)) \in \mathbb{R}^2$$

   lie on a line. Prove that $f''(x) = 0$ for some $x \in I$.

   Solution:

   Let m be the slope of this line. By the Mean Value Theorem, there exists $d \in [a,b] \subseteq I$ such that $f'(d) = (f(b) - f(a))/(b-a) = m$; and there exists $e \in [b,c] \subseteq I$ such that $f'(e) = (f(c) - f(b))/(c-b) = m$. By Rolle's Theorem applied to f', there exists $x \in [d,e] \subseteq I$ such that $f''(x) = 0$.

10. a) Consider the ideal $I = (2x^2 + 2x + 1)$ in $\mathbb{Z}[x]$. Determine whether I is a prime ideal, and whether it is maximal.

b) Is $R = \mathbb{Z}[x]/I$ an integral domain? If your answer is no, find a zero-divisor in R. If your answer is yes, find a complex number α such that the fraction field of R is isomorphic to $\mathbb{Q}[\alpha]$.

Solution:

(a) By the quadratic formula, the polynomial $2x^2 + 2x + 1$ has no roots in $\mathbb{Q}$. Since it has degree 2, it is irreducible over $\mathbb{Q}$, and hence also over $\mathbb{Z}$ since it is primitive. Since $\mathbb{Z}[x]$ is a UFD and $2x^2+2x+1$ is irreducible, the principal ideal I is a prime ideal. This ideal is not maximal, because it is strictly contained in $(2x^2+2x+1, 3)$. (To see that the latter ideal is proper and maximal, note that $2x^2 + 2x + 1$ is irreducible in $\mathbb{F}_3[x]$, being of degree 3 and having no roots; and so $\mathbb{F}_3[x]/(2x^2 + 2x + 1) = \mathbb{Z}[x]/(2x^2 + 2x + 1, 3)$ is a field.)

(b) R is an integral domain because I is a prime ideal. Let $\alpha = (-1 + i)/2 \in \mathbb{C}$, which is a root of $2x^2 + 2x + 1$. Then R is isomorphic to $\mathbb{Z}[\alpha]$, and so the fraction field of R is isomorphic to $\mathbb{Q}[\alpha]$.

11. a) Show that in some open neighborhood of the origin in the (x, y)-plane $\mathbb{R}^2$, there is a differentiable function $z = f(x, y)$ satisfying

$$z^5 - z = x^2 + y^2.$$

b) On a sufficiently small neighborhood of the origin, how many such implicit functions f are there?

c) For each such implicit function f, determine whether the origin is a critical point.

Solution:

(a) At $(x, y) = (0, 0)$, the condition is that $z^5 = z$; i.e., $z \in \{0, 1, -1\}$. Let $F(x, y, z) = z^5 - z - x^2 - y^2$. Then $\partial F/\partial z = 5z^4 - 1$, which does not vanish at any of the above three values of z. So by the Implicit Function Theorem, for each of these three values, there is a differentiable function $z = f(x, y)$ defined in a neighborhood of the origin, whose graph lies on the locus of $z^5 - z = x^2 + y^2$, and for which $f(0, 0)$ is equal to that value of z.

(b) For each choice of $z \in \{0, 1, -1\}$ in part (a), and each sufficiently small neighborhood of the origin, the Implicit Function Theorem says that the function f is unique. So there are exactly three such implicit functions.

(c) For each such implicit function f, we may compute the partial derivatives implicitly: $5z^4\partial z/\partial x - \partial z/\partial x = 2x$ and $5z^4\partial z/\partial y - \partial z/\partial y = 2y$, so $\partial z/\partial x = 2x/(5z^4 - 1)$ and $\partial z/\partial y = 2y/(5z^4 - 1)$. At $(x, y) = (0, 0)$, these are both equal to 0, and so the origin is a critical point for each of the three implicit functions.

12. Let M be the 4×4 real matrix each of whose entries is equal to 1.

a) Find the kernel, image, rank, nullity (dimension of the kernel), trace, and determinant of M.

b) Find the characteristic polynomial of M, the eigenvalues of M, and the dimensions of the corresponding eigenspaces.

c) Determine whether M is diagonalizable.

Solution:

(a) The kernel is $\{(x, y, x, w) \in \mathbb{R}^4 \mid x + y + z + w = 0\}$. So the nullity is 3 and the rank is 1. The image is $\{(x, y, x, w) \in \mathbb{R}^4 \mid x = y = z = w\}$. The trace is 4, being the sum of the diagonal elements. The determinant is 0 since the rank is less than 4.

(b) Since the nullity is 3, there is a 3-dimensional eigenspace with eigenvalue 0. Since the rank is 1, there is a one dimensional eigenspace with non-zero eigenvalue c. Since the sum of the eigenvalues is equal to the trace, $c = 4$. So the characteristic polynomial is $X^3(X - 4)$.

(c) Since the sum of the dimensions of the eigenspaces is the dimension of the vector space $\mathbb{R}^4$, there is a basis of eigenvectors; and so M is diagonalizable.