Midterm Exam III for Math 110, Spring 2015

Solutions
1. (a) Find the general solution to the differential equation

\[ e^{kt} z' = \frac{a}{z} . \]

**Solution:** Move the \( e^{kt} \) to the right-hand side and the \( z \) to the left to see that it is a separable equation:

\[ z z' = a e^{-kt} . \]

Integrate to get \( \frac{1}{2} z^2 = \frac{a}{k} e^{-kt} + c \), and solve for \( z \) yielding

\[ z = \sqrt{C - \frac{2a}{k} e^{-kt}} \]

where this \( C \) is twice the old constant \( c \), therefore can still be any real number.

(b) If the units of \( t \) are seconds and the units of \( z \) are feet, what are the units of \( k \) and \( a \)?

**Solution:** The exponent must be unitless so the time constant \( k \) has units of inverse seconds. Multiplying through by \( z \) we see that \( a = e^{kt} z z' \) which has units of (unitless) times feet times feet per second; therefore the answer is square feet per second.
2. (a) Circle the number of the correct solution to the initial value problem
\[ y' = y \frac{\sin x}{x^2} \; ; \; \; y(2) = 3. \]

(i) \[ y = 3e^{2-x} \]

(ii) \[ y = \ln \left( e^3 + \int_2^x \frac{\sin t}{t^2} dt \right) \]

(iii) \[ y = e^{3+\int_2^x \frac{\sin t}{t^2} dt} \]

(iv) \[ y = 3 \int \frac{\sin x}{x^2} dx \]

(v) \[ y = e^{\int \frac{\sin x}{x^2} dx + C} \]

(vi) \[ y = 3 + e^{\int_2^x \frac{\sin t}{t^2} dt} \]

(vii) \[ y = 3e^{\int_2^x \frac{\sin t}{t^2} dt} \]

Solution: This is separable. The general solution is obtained by integrating \( y'/y = \sin x/x^2 \) to obtain
\[ \ln |y| = \int \frac{\sin x}{x^2} \; dx. \]

Because the initial value occurs at \( x = 2 \) we turn the indefinite integral into a definite integral starting at \( x = 2 \), making sure to add in the arbitrary constant:
\[ \ln |y| = C + \int_2^x \frac{\sin t}{t^2} \; dt. \]

Solving for \( y \) gives
\[ y = C_1 e^{\int_2^x \frac{\sin t}{t^2} dt} \]

where \( C_1 \) is \( \pm e^C \) for the previous constant, \( C \). When \( x = 2 \) the integral is zero and the expression evaluates to \( C_1 e^0 = C_1 \). Therefore \( C_1 = 3 \) and the correct answer is (vii).
(b) Precisely one of the following statements is true; please circle the corresponding number.

(i) \( y(x) \to \infty \) as \( x \to \infty \).

(ii) \( y(x) \) is defined for all \( x \geq 2 \) and has a horizontal asymptote.

(iii) \( y(x) \) is defined for all \( x \geq 2 \) and has a vertical asymptote.

(iv) \( y(x) \) has a vertical asymptote and is not defined past that point.

(v) The solution may have a vertical or horizontal asymptote, depending on the initial condition.

(vi) \( y(x) \to -\infty \) as \( x \to \infty \).

**Solution:** The integral of \( \frac{dy}{y} \) from any positive constant to infinity diverges. The integral of \( \frac{\sin x}{x^2} \) from any positive constant to infinity converges. The blow-up test then shows that \( x \) grows without bound, while \( y \) approaches a limiting value. The answer that describes this is \( (ii) \).
3. A benefactor sets up a trust fund by giving a steady stream of money, with the rate of giving increasing over time according to the formula \( \text{rate} = 100,000t \) per year at time \( t \) years, starting at time \( t = 0 \) years with no money in the account. The account also grows by return on investment at the rate of 2% per year.

(a) Write an initial value problem for the value after time \( t \) years.

**Solution:** Let \( V(t) \) be the value at time \( t \). Then

\[
V' = 0.02V + 100,000t \text{ with } V(0) = 0.
\]

(b) Solve this initial value problem.

**Solution:** This is a linear first order equation: \( V' - 0.02V = 100,000t \). Multiply both sides by the integrating factor \( e^{-0.02t} \) and integrate yielding the indefinite integral solution

\[
e^{-0.02t}V = \int 100,000 t e^{-0.02t} \, dt.
\]

Integrating by parts on the right-hand side gives

\[
e^{-0.02t}V = 5,000,000(t + 50)e^{-0.02t} + C
\]

Solving for \( V \) and writing \( 0.02t \) as \( t/50 \) gives the general solution

\[
V(t) = Ce^{t/50} - 5,000,000 t - 250,000,000.
\]

Setting \( V(0) = 0 \) yields \( C = 250,000,000 \), therefore

\[
V(t) = 250,000,000(e^{t/50} - 1) - 5,000,000 t = 250,000,000 \left( e^{t/50} - 1 - \frac{t}{50} \right).
\]

(c) Write an exact expression for the value after one year.

**Solution:** Plug in \( t = 1 \) to get \( 250,000,000(e^{1/50} - 1 - 1/50) \).

(d) Give an approximate numerical value to the expression in part (c) by using the quadratic Taylor approximation at \( t = 0 \) to the expression computed in part (b) at \( t = 1 \).

**Solution:** The quadratic Taylor approximation to \( e^{t/50} \) is \( 1 + t/50 + t^2/(2 \cdot 50^2) \), leading to

\[
P_2(t) = 250,000,000 \left( 1 + \frac{t}{50} + \frac{t^2}{2 \cdot 50^2} - 1 - \frac{t}{50} \right) = 250,000,000 \frac{t^2}{5,000} = 50,000 t^2.
\]

Plugging in \( t = 1 \) gives $50,000 for the value after one year. Note: this the same amount the benefactor has put in! The effect of the investment return does not show up until you take the cubic Taylor polynomial: the extra term of \( 250,000,000(t/50)^3/3! = 333.33 \) approximates the true interest of $335.00 quite well.
4. A contour plot is shown for the function $u(x, y)$. For each item please circle T for true or F for false.

F: $u$ is changing faster near $(1, 2)$ than it is near $(2, 1)$.

The contours are farther apart near $(1, 2)$ so the function changes more slowly there.

F: It is possible that $u(x, y) = \frac{1}{1 + x + y}$.

Two reasons why not. One is that any function that depends only on $x + y$ would have straight contours sloping at 45 degrees, because those are the level curves of $x + y$. (Same logic as in hwk10 #1.) Another is that $1/(1+x+y)$ is symmetric in $x$ and $y$ so the contours should be symmetric about the line $y = x$.

T: If $u(x, y)$ represents my utility then I am indifferent between the outcomes at $(1, 2)$ and $(2, 1)$.

There is a contour passing through both of these points, therefore the utility value is equal at the two points, which is the definition of indifference.

F: It is possible that $u(x, y) = ye^{-x}$.

This increases when $y$ increases and $x$ decreases, contradicting the fact that there are level curves along which $y$ increases and $x$ decreases.
5. Compute $\int_1^5 \int_0^2 \frac{x + 1}{y} \, dx \, dy$.

**Solution:** The easiest route is to use the magic product formula to see that

$$
\int_1^5 \int_0^2 \frac{x + 1}{y} \, dx \, dy = \left( \int_1^5 \frac{dy}{y} \right) \times \left( \int_0^2 (x + 1) \, dx \right).
$$

Evaluating these gives

$$(\ln 5 - \ln 1) \times \left( 2 + \frac{1}{2} \cdot 2^2 \right) = 4 \ln 5.$$
6. Compute the average temperature over the upper half of the unit disk in the $x$-$y$ plane if the temperature is given by $T(x, y) = 2y$.

**Solution:** The area if the half disk, $R$, is $\pi/2$ so the average is $(2/\pi)$ times the integral of $T$ over $R$. The region $R$, described in horizontal strips is
\[ \{ (x, y) : 0 \leq y \leq 1 \text{ and } -\sqrt{1-y^2} \leq y \leq \sqrt{1-y^2} \} . \]

Therefore the average temperature is
\[ \frac{2}{\pi} \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 2y \, dx \, dy . \]

In the inner integral $y$ is constant and we get $4y\sqrt{1-y^2}$. Thus, the average temperature is given by
\[ \frac{2}{\pi} \int_0^1 4y\sqrt{1-y^2} \, dy . \]

The substitution $u = 1 - y^2$ allows us to evaluate this as
\[ \frac{2}{\pi} \left( -\frac{4}{3} (1 - y^2)^{3/2} \right)_0^1 = \frac{8}{3\pi} . \]
7. A trapezoid $T$ has corners at $(±2, 0)$ and $(±1, 3)$.

(a) Describe the region $T$ in horizontal strips:

$$T = \{(x, y) : \cdots\}.$$ 

**Solution:** $T = \{(x, y) : 0 \leq y \leq 3 \text{ and } -2 + y/3 \leq x \leq 2 - y/3\}$.

(b) If the dart is thrown at the region $T$ and lands in a random location, distributed uniformly over $T$, what density describes this probability distribution?

**Solution:** The area of $T$ is $(1/2)(3)(2 + 4) = 9$. The density of the uniform distribution is a constant which is the reciprocal of the area. Therefore, the density of the uniform distribution on $T$ is the constant $1/9$ over the region $T$.

(c) What is the mean of the $Y$ value chosen by the dart?

**Solution:** The mean is $\int_{T} y \, dA$. As a double integral, using $(1/9) \, dx \, dy$ for $dA$ and the limits of integration given by the answer to part (a), this becomes

$$\int_{0}^{3} \int_{-2+\frac{y}{3}}^{2-\frac{y}{3}} \frac{1}{9} \, dx \, dy.$$ 

The inner integral is $\frac{1}{9} \left(2 - \frac{y}{3}\right)$ and integrating this from 0 to 3 gives $\frac{4}{9}y - \frac{1}{27}y^2$ evaluated between 0 and 3, which comes out to $12/9 - 9/27 = 1$. 
8. Suppose that profit depends on price, cost per unit, and number sold according to the formula $P = n(p - c)$. The number sold is a function of the price.

(a) Make a branch diagram for this.

(b) State which of $P, p, c$ and $n$ are independent variables, which are dependent variables and which are intermediate variables.

**Solution:** Profit, $P$, is the dependent variable. The cost, $c$, and the price, $p$ are independent variables. The number, $n$ is an intermediate variable because it depends on $p$ and $P$ depends on it.

(c) Suppose that the present values of $p, c$ and $n$ are respectively 3, 1 and 10,000. Write a formula for the increase in profit per unit increase in price assuming that cost is held constant. Your formula may contain symbols for functions and derivatives not explicitly known.

**Solution:** By the multivariate chain rule, the rate of change of $P$ with respect to $p$ is obtained by summing products along the $P$-$n$ branch and the $P$-$n$-$p$ branch. This yields

$$\frac{\partial P}{\partial p} = \frac{\partial P}{\partial p} + \frac{\partial P}{\partial n} \frac{dn}{dp}.$$ 

Note the ambiguous use of the term $\partial P/\partial p$ occurring on both sides with different meanings! Computing partial derivatives of $P = n(p - c)$ gives $P_n = p - c$ and $P_p = n$. At the given values, $P_p = 10,000$ and $P_n = 2$, so the right-hand side is $10,000 + 2 \frac{dn}{dp}$. This is the answer sought.
9. A customer’s satisfaction with her hotel room in Barbados is modeled by the utility function 
\[ u = 100 - (T - 70)^2 - \frac{c}{10} \] where \( T \) is temperature in degrees Farenheit and \( c \) is cost per night in dollars. If the room is presently 78° F and costs $200 per night, how many more dollars per night would she be willing to pay per extra degree of lower temperature in the room?

**Solution:** The slope of the tangent line to the level curve of \( u(T, c) \) through the point (78, 200) is \(-u_T/u_c\). The marginal rate of substitution is the negative of this slope, so it is \( u_T/u_c \). Here, \( u_T = -2(T - 70) = -16 \) and \( u_c = -1/10 \), so the slope of the tangent in the \( T-c \) plane is 160. This means she would be willing to pay $160 per night extra per degree cooler at this point.
10. Use the increment theorem for the function \( f(x, y) = \sqrt{x + \ln y} \) to give a numerical estimate of \( \sqrt{99 + \ln 1.3} \).

You don’t need to state the theorem, but if you want to be eligible for partial credit you should state the values of \( \Delta x \) and \( \Delta y \) and show how the relevant partial derivatives are evaluated.

**Solution:** We will approximate near \( x = 100, y = 1 \) with \( \Delta x = -1 \) and \( \Delta y = 0.3 \). Then

\[
\Delta f \approx f_x \Delta x + f_y \Delta y
\]

where \( f_x = 1/(2\sqrt{x + \ln y}) \) which evaluates to \( 1/20 \) and \( f_y = (1/y)/(2\sqrt{x + \ln y}) \) which also evaluates to \( 1/20 \). Thus \( \Delta f \approx (-1 + 0.3)/20 = -0.035 \) leading to

\[
\sqrt{99 + \ln 3} \approx 10 - 0.035 = 9.965.
\]
| 1. $\int k \, dx = kx + C$ (any number $k$) | 12. $\int \tan x \, dx = \ln |\sec x| + C$ |
| 2. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$ | 13. $\int \cot x \, dx = \ln |\sin x| + C$ |
| 3. $\int \frac{dx}{x} = \ln |x| + C$ | 14. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$ |
| 4. $\int e^x \, dx = e^x + C$ | 15. $\int \csc x \, dx = -\ln |\csc x + \cot x| + C$ |
| 5. $\int a^x \, dx = \frac{a^x}{\ln a} + C$ $(a > 0, a \neq 1)$ | 16. $\int \sinh x \, dx = \cosh x + C$ |
| 6. $\int \sin x \, dx = -\cos x + C$ | 17. $\int \cosh x \, dx = \sinh x + C$ |
| 7. $\int \cos x \, dx = \sin x + C$ | 18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) + C$ |
| 8. $\int \sec^2 x \, dx = \tan x + C$ | 19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$ |
| 9. $\int \csc^2 x \, dx = -\cot x + C$ | 20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \left| \frac{x}{a} \right| + C$ |
| 10. $\int \sec x \tan x \, dx = \sec x + C$ | 21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \left( \frac{x}{a} \right) + C$ $(a > 0)$ |
| 11. $\int \csc x \cot x \, dx = -\csc x + C$ | 22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \left( \frac{x}{a} \right) + C$ $(x > a)$ |