Answer Key

Calculus II (Math 114)
Fall 2006

The Makeup Final Exam

There are nine questions on this examination. Some have multiple parts. It is important to show your work and justify each statement. You will receive partial credit for substantial progress towards the answers. You will lose partial credit for answers that are not justified. Please write legibly. No calculators, books, or notes may be used except for one two-sided 8.5x11 sheet of notes.

Name:__________________________

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TA:__________________________

Problem 1. **(10pts)** Find the solution to the equation,

$$y' = \frac{x}{2y\sqrt{x^2-16}}, \quad y(5) = 2.$$

Solution

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx \quad (4\text{ pts})$$

$$y^2 = \sqrt{x^2-16} + c \quad (4\text{pts})$$

$$y(5)=2 \Rightarrow 4 = 3 + c \Rightarrow c = 1 \quad (2\text{ pts})$$

$$\boxed{\therefore\ y^2 = \sqrt{x^2-16}+1}$$

Problem 2. (10pts) (a) The general first-order linear differential equation is

$$\frac{dy}{dx} + P(x)y(x) = Q(x).$$

Under what conditions on $Q(x)$ and/or $P(x)$ is this equation separable?
(b) Suppose that $y_p(x)$ is a solution to the differential equation of the first part, and that $y_c(x)$ is a solution to the complementary equation

$$\frac{dy}{dx} + P(x)y(x) = 0.$$

Show that $y_p(x) + Cy_c(x)$ is a solution to the differential equation of the first part, for any constant C.
(c) Solve the initial-value problem

$$\frac{dy}{dx} + x^2y(x) = x^2, \quad y(0) = 0.$$

Solution

(2 pts) a) IF $P(x) = 0$, it is separable.

(3 pts) b) $y_p'(x) + P(x)y_p(x) = Q(x)$

$y_c'(x) + P(x)y_c(x) = 0 \Rightarrow Cy_c'(x) + CP(x)y_c(x) = 0$

$\cancel{y_p'(x) + y_c'(x) + P(x)y_p(x) + P(x)y_c(x) = Q(x)}$

$\cancel{(y_p' + y_c') + P(y_p + y_c) = Q}$

Adding $\Rightarrow$ $y_p' + Cy_c' + Py_p + CPy_c = Q$

$\Rightarrow (y_p' + Cy_c') + P \cdot (y_p + Cy_c) = Q$

$\Rightarrow (y_p + Cy_c)' + P(y_p + Cy_c) = Q$

c) (on next page)

Solution (cont.)

(5 pts) c) Int. factor $= e^{\int x^2 dx} = e^{\frac{1}{3}x^3}$ (1 pt)

$$e^{\frac{1}{3}x^3}\frac{dy}{dx} + x^2 e^{\frac{1}{3}x^3} y = x^2 e^{\frac{1}{3}x^3}$$

$$\frac{d}{dx}\left(e^{\frac{1}{3}x^3} y\right) = x^2 e^{\frac{1}{3}x^3}$$

$$e^{\frac{1}{3}x^3} y = \int x^2 e^{\frac{1}{3}x^3} dx \qquad (2 \text{ pts})$$

$$e^{\frac{1}{3}x^3} y = e^{\frac{1}{3}x^3} + c$$

$$y = 1 + ce^{-\frac{1}{3}x^3} \qquad (1 \text{ pt})$$

$$y(0)=0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$$

$$\boxed{y = 1 - e^{-\frac{1}{3}x^3}} \qquad (1 \text{ pt})$$

Problem 3. **(10pts)** (a) Solve the homogeneous equation

$$y'' + 2y' + 2y = 0, \quad y(0) = 1, \; y'(0) = 1.$$

(b) Find the general solution to

$$y'' + 2y' + 2y = x^2$$

Solution

(5pts) a) $y = e^{rx} \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$$\boxed{\therefore y = e^{-x}(c_1 \cos x + c_2 \sin x)}$$

(5 pts) b) $y = ax^2 + bx + c \Rightarrow y' = 2ax + b \Rightarrow y'' = 2a$

$$y'' + 2y' + 2y = 2a + 4ax + 2b + 2ax^2 + 2bx + 2c$$

$$= 2ax^2 + (4a + 2b)x + (2a + 2b + 2c)$$

$$y'' + 2y' + 2y = x^2 \Rightarrow 2a = 1, \; 4a + 2b = 0, \; 2a + 2b + 2c = 0$$

$$\Rightarrow a = \frac{1}{2}, \; b = -1, \; c = \frac{1}{2}$$

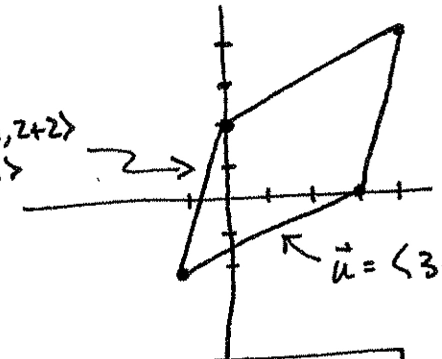
$$\boxed{\therefore y = \frac{1}{2}x^2 - x + \frac{1}{2} + e^{-x}(c_1 \cos x + c_2 \sin x)}$$

Problem 4. **(10pts)** (a) Find the area of the parallelogram which has the points $(-1,-2)$, $(0,2)$, $(4,4)$ and $(3,0)$ as its vertices. (b) Let P be a parallelogram in the plane with all four sides lengths equal. Use vectors to show that the two diagonals of P are perpendicular.

Solution

pts) a)

$\vec{v} = \langle 0+1, 2+2 \rangle = \langle 1,4 \rangle$

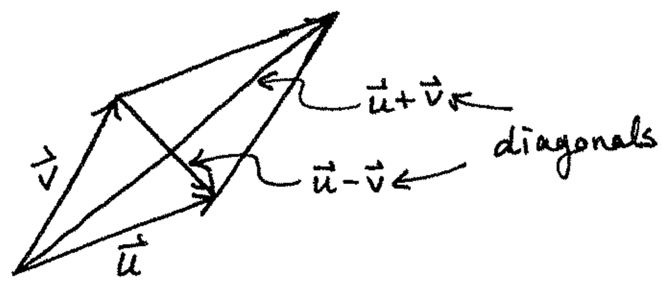


$\vec{u} = \langle 3+1, 0+2 \rangle = \langle 4,2 \rangle$

$$\boxed{\text{Area} = \left|\begin{matrix} 4 & 2 \\ 1 & 4 \end{matrix}\right| = 16-2 = 14}$$

pts) b)

$|\vec{u}| = |\vec{v}|$



$$(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{v}$$
$$= |\vec{u}| - |\vec{v}|$$
$$= 0$$

Problem 5. **(10pts)** Consider the circular helix

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$$

(a) Find the arc length function $s = s(t) = \int_0^t \|\vec{r}\,'(u)\| du$ as a function of t.

(b) Parameterize $\vec{r}$ with respect to s (i.e. determine $\mathbf{R}(s) = \vec{r}(t(s))$).

(c) Find the coordinates of the point Q on the helix such that the arc-length from $P = (3, 0, 0)$ to Q is 5π.

(d) Find a curvature $\kappa(s)$ of the helix.

Solution

(3 pts) a) $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 4\rangle$

$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$

$\boxed{s(t) = \int_0^t 5\,dt = 5t}$

(2 pts) b) $\Rightarrow t = \frac{1}{5}s$

$\therefore \vec{r}(s) = \langle 3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4s}{5}\rangle$

(2 pts) c) $\vec{r}(0) = \langle 3,0,0\rangle$

Arc length from $P = s(t) = 5t = 5\pi \Rightarrow t = \pi$

$\boxed{\vec{r}(\pi) = \langle 3\cos\pi, 3\sin\pi, 4\pi\rangle = \langle -3, 0, 4\pi\rangle}$

(3 pts) d) $\vec{T}(s) = \langle -\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}\rangle$

$K(s) = \left|\frac{d\vec{T}}{ds}\right|$

$= \left|\langle -\frac{3}{25}\cos\frac{s}{5}, -\frac{3}{25}\sin\frac{s}{5}, 0\rangle\right|$

$\boxed{K = \frac{3}{25}}$

Problem 6. (10pts) A ball is thrown into the air from the origin in xyz-space (the xy-plane represents the ground and the positive y-axis points north). The initial velocity vector of the ball is

$$\vec{v}_0 = \langle 0, 80, 80 \rangle.$$

The spin of the ball causes an eastward acceleration of 2 ft/s^2 in addition to gravitational acceleration. Thus the acceleration vector produced by the combination of gravity and spin is

$$\vec{a} = \langle 2, 0, -32 \rangle.$$

First find the velocity vector $\vec{v}(t)$ of the ball and its position vector $\vec{r}(t)$. Then determine where and with what speed the ball hits the ground.

Solution

$\vec{V}_0 = \langle 0, 80, 80 \rangle$

$\vec{a} = \langle 2, 0, -32 \rangle$

$\vec{a}(t) = \vec{v}'(t) \Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \langle 2t, 0, -32t \rangle + \langle 0, 80, 80 \rangle$

(3pts) $\boxed{\vec{v}(t) = \langle 2t, 80, -32t + 80 \rangle}$

$\vec{v}(t) = \vec{r}'(t) \Rightarrow \vec{r}(t) = \int v(t)\,dt = \langle t^2, 80t, -16t^2 + 80t \rangle + \langle 0, 0, 0 \rangle$

(3pts) $\boxed{\vec{r}(t) = \langle t^2, 80t, -16t^2 + 80t \rangle}$

Ball hits the ground when z-component of $\vec{r}(t) = 0$.

That is, when $-16t^2 + 80t = 0 \Rightarrow 80 = 16t \Rightarrow t = 5$ sec.

(2pts) $\boxed{\vec{r}(5) = \langle 25, 400, 0 \rangle}$ ← ball hits the ground 400ft north and 400ft east

$\vec{v}(5) = \langle 10, 80, -80 \rangle$

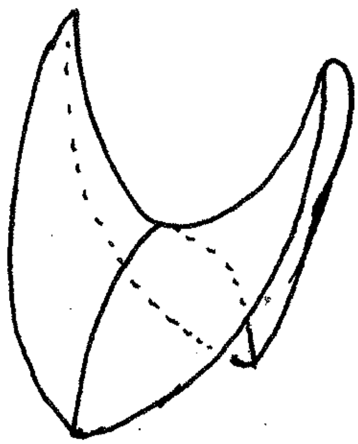
$|\vec{v}(5)| = 10\sqrt{1 + 64 + 64} = \boxed{10\sqrt{129}\ \text{ft/s}}$ ← speed that ball hits the ground

(2pts)

Problem 7. **(10pts)** Let $f(x,y) = xy$ be defined on the domain $\Omega = \{(x,y) : x^2 + y^2 \leq 4\}$.
(a) Sketch Ω. Is it closed, open or neither ?
(b) Determine critical points in the interior of Ω, and classify them as: *local minima, maxima or saddle points.*
(c) Apply the Lagrange multipliers method to find critical points on the boundary of Ω: $\{(x,y) : x^2 + y^2 = 4\}$.
(d) Determine absolute minima and maxima of f on Ω, or show that they do not exist.

Solution

(2 pts) a)



(2pts) d) Absolute minima $= -2 = f(\sqrt{2},-\sqrt{2}) = f(-\sqrt{2},\sqrt{2})$
Absolute maxima $= 2 = f(\sqrt{2},\sqrt{2}) = f(-\sqrt{2},-\sqrt{2})$

(3pts) b) $f_x(x,y) = y$, $f_y(x,y) = x$ $\Big\}$ $\boxed{\text{Critical Point : } (0,0) \quad \underline{\text{Saddle}}}$

(3pts) c) $f(x,y) = xy, \quad g(x,y) = x^2 + y^2 = 4$

$\nabla f = \langle y, x \rangle, \quad \nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \underbrace{y = 2\lambda x, \quad x = 2\lambda y}, \quad x^2 + y^2 = 4$

$\Rightarrow x = 4\lambda^2 x \Rightarrow x = 0$ or $1 = 4\lambda^2 \Rightarrow \lambda = \pm\frac{1}{2}$

$\cancel{x = 0 \text{ or } x = \frac{1}{4\lambda^2} \Rightarrow y = 0 \text{ not on boundary}, \ y = \frac{1}{2\lambda}, \ \left(\frac{1}{4\lambda^2}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 4 \Rightarrow \frac{1}{16\lambda^4} + \frac{1}{4\lambda^2} = 4 \Rightarrow 1 + 4\lambda^2 = 64\lambda^4}$

$x = 0 \Rightarrow y = 0$ ∴ not on boundary
$\lambda = \frac{1}{2} \Rightarrow x = y \Rightarrow x = y = \pm\sqrt{2}$
$\lambda = -\frac{1}{2} \Rightarrow x = -y \Rightarrow x = -y = \pm\sqrt{2}$ $\Big\}$ Crit pts on boundary

Problem 8. (10pts) Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{z}\, dz\, dy\, dx.$$

Solution

$$\iiint_E \sqrt{z}\, dV \qquad E = \{ (x,y,z) \mid x^2+y^2+z^2 \le 1 \text{ and } x \ge 0, y \ge 0, z \ge 0 \} \quad \text{1st octant}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \sqrt{\rho\cos\phi} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} d\theta}_{} \cdot \underbrace{\int_0^{\pi/2} (\cos\phi)^{1/2} \cdot \sin\phi\, d\phi}_{} \cdot \underbrace{\int_0^1 \rho^{5/2}\, d\rho}_{\frac{2}{7}\rho^{7/2} \Big|_0^1}$$

$$= \theta \Big|_0^{\pi/2}$$

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$-\int u^{1/2}\, du = -\frac{2}{3}u^{3/2}$

$-\frac{2}{3}(\cos\phi)^{3/2} \Big|_0^{\pi/2}$

$$= \left[\frac{\pi}{2} - 0\right] \cdot -\frac{2}{3}[0-1] \cdot \frac{2}{7}[1-0]$$

$$= \left(\frac{\pi}{2}\right) \cdot \left(\frac{2}{3}\right) \cdot \left(\frac{2}{7}\right)$$

$$= \boxed{\frac{2\pi}{21}}$$

1 pt. for each correct set of bounds

3 pts for $\rho^2 \sin\phi$

1 pt. for $\sqrt{\rho\cos\phi}$

3 pts for correct answer

Problem 9. (10pts) Set

$$g(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Show that the partial deriviatives g_x and g_y both exist at $(0,0)$. What are their values at $(0,0)$,

(b) Show that $\lim_{(x,y)\mapsto(0,0)} g(x,y)$ does not exist.

Solution

(6pts) a) $g_x(0,0) = \lim_{h\to 0} g(h,0) = \lim_{h\to 0} \frac{h^2\cdot 0^2}{h^4+0^4} = 0$

$g_y(0,0) = \lim_{h\to 0} g(0,h) = \lim_{h\to 0} \frac{0^2\cdot h^2}{0^4+h^4} = 0$

(4pts) b) $\lim_{x\to 0} g(x,0) = \lim_{x\to 0} \frac{x^2\cdot 0^2}{x^4+0^4} = 0$

$\lim_{x\to 0} g(x,x) = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$

$\neq$ $\therefore$ limit does not exist