Final Exam
May 1, 2013

Name: ____________________________
Penn ID #: ____________________________

Show all your work. A correct answer without supporting work receives little or no credit!

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1. Consider the surface \( z = f(x, y) = 2x^2 + y^2 \). Find the tangent plane to the surface at the point \((x, y, z) = (1, -1, 3)\) and find where this plane intersects the \(y\)-axis.

\[
\begin{align*}
(A) & -1 \\
(B) & 1 \\
(C) & -\frac{3}{2} \\
(D) & \frac{3}{2} \\
(E) & 0 \\
(F) & -\frac{1}{2} \\
(G) & \frac{1}{2} \\
(H) & \text{none of these}
\end{align*}
\]
2. Find the point on the plane $2x + y + 2z = 9$ closest to the origin.

(A) $(7, -5, 7)$  (B) $(6, -3, 6)$  (C) $(5, -1, 5)$  (D) $(1, 5, 1)$

(E) $(1, 3, 2)$  (F) $(-1, 9, 1)$  (G) $(2, 1, 2)$  (H) $(0, 9, 0)$
3. \( \int_0^1 \int_{2y}^2 4\sin(x^2) \, dx \, dy = \)

(A) \( 1 - \frac{1}{4}\cos4 \) \hspace{1cm} (B) \( 1 + \cos4 \) \hspace{1cm} (C) \( 1 + \frac{1}{4}\cos4 \) \hspace{1cm} (D) \( -1 - \frac{1}{2}\cos4 \)

(E) \( -1 + \frac{1}{2}\cos4 \) \hspace{1cm} (F) \( 1 - 2\cos4 \) \hspace{1cm} (G) \( 1 + \frac{1}{2}\cos4 \) \hspace{1cm} (H) \( 1 - \cos4 \)
4. The function \( f(x, y) = x^2 - 4x + y^3 - 3y \) has two critical points. One of them is a saddle point, whose coordinates are \((a, b)\). Find the difference of the coordinates \(a - b\).

(A) 1  (B) -1  (C) 2  (D) -2
(E) 3  (F) -3  (G) \(-\frac{1}{2}\)  (H) \(\frac{1}{2}\)
5. Let \( f(x, y) = \sqrt{x^2 - y^2} \). Note \( f(5, 3) = 4 \). Using differentials to approximate \( f(4.9, 3.2) \) we get:

- (A) \( 4 - \frac{1}{20} \)
- (B) \( 4 + \frac{1}{10} \)
- (C) \( 4 - \frac{1}{10} \)
- (D) \( 4 + \frac{1}{20} \)
- (E) \( 4 - \frac{1}{20} \)
- (F) \( 4 - \frac{1}{10} \)
- (G) \( 4 \)
- (H) none of these
6. Consider three events $E$, $F$, and $G$. Assume that they have the following probabilities: $\Pr(E) = 1/2$, $\Pr(F) = 2/5$, and $\Pr(G) = 2/5$. Assume we also know that $\Pr(E|F) = 2/3$, $\Pr(E|G) = 5/8$, $\Pr(F|G) = 3/8$, and $\Pr((E \cap G)|F) = 1/4$. Then $\Pr(F \cup (E \cap G))$ is:

(A) $11/20$  
(B) $3/10$  
(C) $9/20$  
(D) $1/2$

(E) $3/5$  
(F) $7/10$  
(G) $13/20$  
(H) none of these
7. There are three coins in a fountain. The coins all look the same. Coin A has a probability of $\frac{2}{3}$ of producing a heads, coin B is fair, and coin C has a probability of $\frac{1}{3}$ of producing a heads. One coin is picked at random and flipped twice, producing a heads followed by a tails. What is the probability that it is the coin A?

(A) $\frac{3}{5}$ (B) $\frac{11}{25}$ (C) $\frac{1}{3}$ (D) $\frac{8}{25}$
(E) $\frac{4}{9}$ (F) $\frac{2}{5}$ (G) $\frac{2}{3}$ (H) none of these
8. Coin $A$ produces heads $1/4$ of the time and tails $3/4$ of the time. Coin $B$ is fair. Each coin is tossed twice. What is the probability that coin $B$ produces more heads than coin $A$?

(A) $1/8$  
(B) $3/16$  
(C) $1/4$  
(D) $5/16$

(E) $3/8$  
(F) $11/32$  
(G) $7/16$  
(H) none of these
9. A pair of fair dice with sides numbered 1 through 6 are rolled 12 times. For each roll the sum of the dice is noted. Find the probability that exactly four times a “3” or a “9” is rolled.

(A) $90 \cdot \frac{5^8}{6^{12}}$  
(B) $99 \cdot \frac{5^8}{6^{12}}$  
(C) $110 \cdot \frac{5^9}{6^{12}}$  
(D) $90 \cdot \frac{5^9}{6^{12}}$

(E) $99 \cdot \frac{5^9}{6^{12}}$  
(F) $99 \cdot \frac{5^9}{6^{12}}$  
(G) $110 \cdot \frac{5^9}{6^{12}}$  
(H) none of these
10. Six dice numbered (1, 2, 3, 4, 5, 6) are tossed. Given the dice are fair and independent, find the variance of the sum of the dice.

(A) 19  (B) 13  (C) 15.5  (D) 10
(E) 13.5  (F) 16  (G) 17.5  (H) 18
11. Suppose $X$ and $Y$ are independent continuous random variables uniformly distributed on the intervals $0 \leq x \leq 4$ and $0 \leq y \leq 2$, respectively. Compute the variance of $\sqrt{2}Y - \frac{1}{2}X$. Hint: First find the variance of $X$ and the variance of $Y$.

(A) 1  (B) 1/9  (C) 4/9  (D) 2/3
(E) 1/4  (F) 3/4  (G) 4/3  (H) 1/3
12. Suppose that a random variable $X$ is uniformly distributed on the interval $[1, 6]$. Compute the expected value of $1/(X + 1)$.

(A) $1/7$  
(B) $1/5$  
(C) $1/\ln(7/2)$  
(D) $\ln(7/2)/6$

(E) $\ln(7)/5$  
(F) $5/2$  
(G) $7/2$  
(H) $\ln(7/2)/5$
13. The number of clicks of a Geiger counter is a Poisson process with a mean of one click every 30 seconds. In a given period of one particular minute it is known that there at least 3 clicks. What is the probability that the number of clicks during that minute was at most 4?

(A) $\frac{2}{5}$  
(B) $2e^{-2}$  
(C) $\frac{2e^{-2}}{1-5e^{-2}}$  
(D) $1-5e^{-2}$

(E) $\frac{1-5e^{-2}}{2e^{-2}}$  
(F) $3e^{-2}$  
(G) $\frac{2}{5}e^{-2}$  
(H) none of these
14. The waiting time for an elevator is an exponentially distributed random variable with mean 3 minutes. When your guest arrives at your floor you ask if she had to wait longer than 6 minutes for the elevator. She says no. What is the probability that she had to wait at least 3 minutes? In order to get credit you must set up and evaluate the integral.

(A) $1 - 2e^{-2}$  
(B) $1 - \frac{1}{2}e^{-1}$  
(C) $\frac{1}{2}e^{-1}$  
(D) $\frac{1}{2}e^{-2}$  
(E) $1 - \frac{1}{2}e^{-1/2}$  
(F) $\frac{e^{-1} - e^{-2}}{1 - e^{-2}}$  
(G) $-e^{-1}$  
(H) $\frac{e^{-1} - e^{-2}}{e^{-2} - 1}$
15. Find the value of the constant $k$ for which the following system of linear equations

$$\begin{cases} 
  x + 2y + z = 5 \\
  2x - 2y + 2z = 2 \\
  2x + y - kz = 3 
\end{cases}$$

has exactly one solution.

(A) $k = -3$ (B) $k \neq 2$ (C) $k = 2$ (D) $k \neq -3$

(E) $k \neq 3$ (F) $k = 3$ (G) $k = -2$ (H) none of these
16. We know \( B = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \) and \( AB = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \). Find \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and compute the sum of its entries \( S = a + b + c + d \).

(A) 5/2  (B) −7/2  (C) 5  (D) 7  
(E) 15/2  (F) 4  (G) −6  (H) 9/2
17. Aidan, Julian, and Katya are playing catch. Aidan is three times more likely to throw to Katya than he is to throw to Julian. Julian is three times more likely to throw to Katya than he is to throw to Aidan. Katya is as likely to throw to Aidan as she is to throw to Julian. What is the probability that Katya will have the ball in the long run? In order to get credit you must set up the transition matrix and solve the corresponding linear system.

(A) 1/7  
(B) 2/7  
(C) 1/2  
(D) 2/5  
(E) 1/3  
(F) 2/3  
(G) 3/7  
(H) none of these
18. A simple economy consists of two sectors, I and II. To produce a $1 worth of product in either sector requires 20 cents spent in sector I and 30 cents spent in sector II. How should the production levels be set in order to meet a demand for $20 million worth of product I and $10 million worth of product II? The production level in millions=

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19. Two rods A and B are to be welded end to end to make a 5 meter rod. The length of each of the rods is a normal distributed random variable with means $\mu$ and standard deviations $\sigma$ given in the table below. What is the probability that the welded rods will be within 2 centimeters of 5 meters, i.e., $\Pr(|X_A + X_B - 5 \text{ m}| < 2 \text{ cm})$. Note 1 centimeter $= 10^{-2}$ meters.

\[ A : \mu = 2 \text{ meters}, \sigma = 1 \text{ centimeters} \]

\[ B : \mu = 3 \text{ meters}, \sigma = \sqrt{3} \text{ centimeters} \]

Circle the closest answer. Indicate what you looked up and how you used it.

(A) 95%  (B) 80%  (C) 70%  (D) 50%

(E) 40%  (F) 25%  (G) 15%  (H) 5%
### Standard Normal Distribution Table


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Answer key: CGHE HADH FGAH CFHC GEC