Final exam, Math 240: Calculus III
April 29, 2005

No books, calculators or papers may be used, other than a hand-written note card at most 5" × 7" in size.

For this web version, answers are at the end of the exam.

This examination consists of eight (8) long-answer questions and four (4) multiple-choice questions. Each problem is worth ten points. Partial credits will be given only for long-answer questions, when a substantial part of a problem has been worked out. Merely displaying some formulas is not sufficient ground for receiving partial credits.

- Your name, printed:
- Your Penn ID (last 4 of the middle 8 digits):
- Your signature:
- Your lecture section (circle one):

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Part I. Long-answer Questions.

1. Compute $\det(A^3)$, where $A$ is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}.$$ 

2. Let $C$ be the oriented curve

$$C = \{(x, y) : 4x^2 + 9y^2 = 36, \ x \geq 0, \ y \geq 0\}$$

from $(3, 0)$ to $(0, 2)$. Compute the line integral

$$\int_C (x + 1) \, dy + y \, dx .$$

3. Let $D$ be the cube

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1\}$$

and let $S = \partial D$ be the boundary surface of $D$, oriented by the unit normal vector field $\vec{n}$ on $S$ pointing away from $D$. Compute the oriented surface integral

$$\iint_S (x^2 \vec{i} + xyz \vec{j} + z^3 \vec{k}) \cdot \vec{n} \, dS .$$

4. Let $S$ be the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \ z \geq 0\} ,$$

the upper half of the unit sphere centered at the origin, oriented by the unit normal vector field $\vec{n} = x \vec{i} + y \vec{j} + z \vec{k}$ on $S$. Compute the surface integral

$$\iint_S (x \vec{i} - y \vec{j} + z \vec{k}) \cdot \vec{n} \, dS .$$

5. Let $C$ be the boundary of the rectangle with vertices $(3, 2)$, $(-5, 2)$, $(-5, -7)$ and $(3, -7)$, oriented counter-clockwise. Compute the line integral

$$\oint_C y \, dx - x \, dy \over x^2 + y^2 .$$
6. Let \( y(t) \) be a function which satisfies the differential equation
\[
y''(t) + (1 + t)y'(t) + (1 + t + t^2)y(t) = 0
\]
y(0) = 1, \( y'(0) = 0 \). Determine the values of \( y''(0) \) and \( y'''(0) \).

7. Suppose that a vector-valued function \( \vec{x}(t) \) satisfies the following system of ordinary differential equations
\[
\vec{x}'(t) - A \cdot \vec{x}(t) = \vec{0}, \quad \text{where } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},
\]
and \( \vec{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \). Determine the function \( \vec{x}(t) \) explicitly.

8. Find one (“particular”) solution of the system of differential equations
\[
\begin{cases}
\frac{dx}{dt} + y = t \\
\frac{dy}{dt} - 2x = 0
\end{cases}
\]
In other words, find a pair of real-valued functions \( (x(t), y(t)) \) satisfying the above system of equations. (There are many such solutions.)
[Hint: Try to replace the above system by a single differential equation, then try to find a particular solution of that equation.]

Part II begins on the next page
Part II. Multiple Choice Questions. Please circle your answer.

9. Consider the following matrices

\[ A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \ A_3 = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}, \ A_4 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \]

Which ones can be diagonalized over the real numbers? (In other words, there exists an invertible matrix \( P \) with coefficients in real numbers such that \( P^{-1} \cdot A_i \cdot P \) is a diagonal matrix.)

A. \( A_1 \) and \( A_3 \) only  \hspace{1cm} B. \( A_2, A_3 \) and \( A_4 \) only  \hspace{1cm} C. \( A_3 \) and \( A_4 \) only
D. \( A_2 \) and \( A_4 \) only  \hspace{1cm} E. \( A_1, A_3 \) and \( A_4 \) only  \hspace{1cm} F. \( A_1 \) and \( A_4 \) only
G. \( A_1, A_2, A_3 \) and \( A_4 \)

10. Let \( A \) be a symmetric \( 4 \times 4 \) matrix with real entries. Consider the following statements.

I. \( A \) must have four distinct eigenvalues.
II. There exists an invertible matrix \( C \) with real entries such that \( C \cdot A \cdot C^{-1} \) is a diagonal matrix.
III. The four roots of the characteristic polynomial of \( A \) are all real numbers.
IV. \( A^2 \) is a symmetric matrix.

Which ones among the above statements are true?

A. I, II, III only.  \hspace{1cm} B. II, III, IV only.  \hspace{1cm} C. I, III, IV only.
D. III and IV only.  \hspace{1cm} E. II and III only.  \hspace{1cm} F. II and IV only.
G. I, II, III, IV are all true.  \hspace{1cm} H. None of the above.
11. Suppose that a function \( x(t) \) satisfies the differential equation
\[
t^2 \frac{d^2 x}{dt^2} - 2t \frac{dx}{dt} + 2x(t) = 0,
\]
and \( x(1) = 3, \frac{dx}{dt}(1) = 1 \). What is the value of \( x(2) \)?

A. 0     B. −1     C. 3     D. −4     E. 2     F. 1     G. 5     H. None of the above.

12. Suppose that a function \( y(t) \) satisfies the differential equation
\[
y''(t) + 2y'(t) + y(t) = e^{-2t},
\]
and \( y(0) = y'(0) = 0 \). What is the value of the Laplace transform \( \mathcal{L}\{y(t)\}(s) \) of \( y(t) \) at \( s = 1 \)?

A. \( \frac{1}{12} \)     B. \( \frac{1}{6} \)     C. \( \frac{1}{24} \)     D. \( \frac{1}{4} \)     E. \( \frac{1}{3} \)     F. \( \frac{1}{36} \)     G. None of the above.

Answers:

1. 1728
2. 2
3. \( \frac{5}{2} \)
4. \( \frac{2\pi}{3} \)
5. \( −2\pi \)
6. \( y''(0) = −1, \ y'''(0) = 0 \)
7. \( \vec{x}(t) = \begin{bmatrix} −e^t + 2te^t \\ 2e^t \end{bmatrix} \)
8. \( x = −\frac{3}{2}t − 4, \ \ y = t + \frac{1}{2} \) [Take the derivative of the first equation; then use the second].
9. Only \( A_3 \) and \( A_4 \) can be diagonalized by \textit{real} matrices.
10. Only II, III, and IV are true.
11. \( x(t) = 5t − 2t^2, \ x(2) = 2 \)
12. 