Math 240  FINAL EXAM     May 4, 2010

Circle one:  Professor Ziller
Professor Zywina

Name: ________________________________

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Signature: ______________________________

TA: ______________________________

Recitation Day and Time: ______________________________

You need to show your work, even for multiple choice problems. A correct answer with no work will get 0 points. The only exception are True/False problems, where no work needs to be shown. Each problem is worth 10 points.

You are NOT allowed to use a calculator. The extra double sided sheet of paper needs to be hand written in your own hand writing (no copies allowed).

(Do not fill these in; they are for grading purposes only.)

1) 9)
2) 10)
3) 11)
4) 12)
5) 13)
6) 14)
7) 15)
8)

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Total
1. For what values of $a$ does the following system have infinitely many solutions?

$$
\begin{align*}
    x + y + z &= 1 \\
    2x + ay + z &= 0 \\
    -x + y + z &= 3
\end{align*}
$$

Answer:
(a) 0    (b) 3    (c) 2    (d) 1    (e) 4    (f) $-8$
2. (a) Find the rank of the matrix

\[ A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

(b) What is \( \det A \)?

(c) The set of solutions of the homogeneous system \( AX = 0 \) depends on how many arbitrary parameters?
3. Let $A$ be the matrix

$$A = \begin{pmatrix} -5 & -4 \\ 8 & k \end{pmatrix}$$

For which value of $k$ does there exist an invertible matrix $P$ such that

$$PAP^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

Answer:
(a) 2  (b) 7  (c) $-4$  (d) 8  (e) $-2$  (f) $-8$
4. Find the maximal number of linearly independent eigenvectors for the matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 0 & 4 \\
0 & -1 & 4
\end{pmatrix}
\]
5. Suppose that a pair of solutions \((x(t), y(t))\) to the system of differential equations

\[
\begin{align*}
x' &= -x + ky \\
y' &= y
\end{align*}
\]

satisfies \(\lim_{t \to +\infty} \frac{x(t)}{y(t)} = 1\) where \(k\) is some unknown constant. What is \(k\)?

Answer:
(a) 2  (b) 4  (c) -4  (d) 8  (e) -2  (f) -8
6. Find the general solution to the system

\[ X' = \begin{pmatrix} 2 & -4 \\ 0 & 2 \end{pmatrix} X \]
7. Let \( y(t) \) be the solution of
\[
y'' - 2y' + y = 2e^t
\]
which satisfies the initial conditions \( y(0) = 0 \) and \( y'(0) = 2 \). Find \( y(1) \).

Answer:
(a) \( e \)  (b) \( 1/e \)  (c) \( 3e \)  (d) \( 3/e \)  (e) \( e^2 \)  (f) \( -8 \)
8. Find the solution of the differential equation

\[ y'' + 2y' + 5y = 0 \]

subject to the initial conditions \( y(0) = 2 \) and \( y'(0) = 2 \).
9. Find the solution $y(x)$ of
\[ x^2 y'' + xy' - y = 0 \]
for which $y(1) = 2$ and $y'(1) = 4$. 
10. Find the recurrence relationship for a power series solution of the differential equation

$$(x^2 + 1)y'' + xy' - y = 0$$

at $x = 0$. Determine the first 4 non-zero terms of the series for the solution with $y(0) = 1$, $y'(0) = 2$. 
11. Find the work done by the force field

\[ \mathbf{F}(x, y, z) = e^y \mathbf{i} + (xe^y + e^z) \mathbf{j} + ye^z \mathbf{k} \]

in moving a particle from \((1, 0, 0)\) to \((0, 1, \pi)\) along the helix \(x = \cos(t), y = \sin(t), z = t\).

Answer:
(a) \(e^\pi\)  \hspace{0.5cm} (b) \(e^\pi - 2\)  \hspace{0.5cm} (c) \(e^\pi + 1\)  \hspace{0.5cm} (d) \(e^\pi - 1\)  \hspace{0.5cm} (e) \(2e^\pi - 1\)  \hspace{0.5cm} (f) \(2e^\pi - 3\)
12. Let $C$ be the curve that is the intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counter-clockwise as viewed from above. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x, y, z) = -yx^2 \mathbf{i} + y^2 z \mathbf{j} + z^2 \mathbf{k}.$$

Answer:
(a) 0  (b) $\frac{1}{2}$  (c) $\frac{1}{3}$  (d) $-2$  (e) $-\frac{1}{4}$
13. Let \( \mathbf{F}(x, y) = (y^2, 3xy) \) be a vector field in the plane and let \( C \) be the closed curve shown in the following picture with a counter-clockwise orientation [the curve \( C_1 \) and \( C_3 \) travel along a circle of radius 2 and 1, respectively]. Evaluate the line integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \).

Answer:
(a) \(-10\)  (b) \(\frac{14}{3}\)  (c) \(\frac{10}{3}\)  (d) \(-\frac{10}{3}\)  (e) \(\frac{3}{2}\)
14. Find the outward flux \( \iint_S \mathbf{F} \cdot \mathbf{n} \, d\mathbf{S} \) of the vector field \( \mathbf{F} = 3xy^2 \mathbf{i} + 3yz^2 \mathbf{j} + 3zx^2 \mathbf{k} \) where the surface \( S \) is the boundary of the region \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).
15. This is the only problem where no work needs to be shown. Each answer is worth one point.

Which one of the following statements are true or false:

(a) If det\((A - I) = 0\), then 1 is an eigenvalue of \(A\).  True\(\bigcirc\) False\(\bigcirc\)

(b) If det \(A = 0\), then the system \(AX = 0\) has a unique solution. True\(\bigcirc\) False\(\bigcirc\)

(c) \((AB)^T = A^T B^T\) for all square matrices \(A, B\). True\(\bigcirc\) False\(\bigcirc\)

(d) If \(S\) is any closed surface, then \(\int \int_S \text{curl}\boldsymbol{F} \cdot \boldsymbol{n} dS = 0\). True\(\bigcirc\) False\(\bigcirc\)

(e) If \(\boldsymbol{F} = \nabla f\) then \(\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0\) for all closed curves \(C\). True\(\bigcirc\) False\(\bigcirc\)