Math 240 Final Exam, spring 2011

Name (printed):

TA:

Recitation Time:

This examination consists of ten (10 problems). Please *turn off all electronic devices*. You may use both sides of a 8.5×11 sheet of paper for notes while you take this exam. No calculators, no course notes, no books, no help from your neighbors. **Show all work**, even on multiple choice or short answer questions—the grading will be bases on your work shown as well as the end result. Please **fill in your final answer in the underlined space** in each problem. Remember to put your name at the top of this page. Good luck.

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

Problem	Score (out of)
1	(10)
2	(10)
3	(10)
4	(10)
5	(10)
6	(10)
7	(10)
8	(10)
9	(10)
10	(10)
Total	(100)

1. (10 pts) (a) Give an example of a 3×3 matrix A which has only two distinct eigenvalues and A is *not* diagonalizable. In other words, there does not exist an invertible 3×3 matrix C such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix. **Justify your answer**.

$A =$ ____________________

(b) Give an example of a 3×3 matrix B which has only two distinct eigenvalues and B is diagonalizable. **Justify your answer**.

$B =$ ____________________

2. (10 pts) Find A^{201} if

$$A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$$

$A^{201} =$ ________________.

3. (10 pts) Give an example of a homogeneous linear ordinary differential equation which is *not ordinary* at $x = 0$ and has a regular singular point at $x = 0$, which has two linearly *independent* solutions of the form

$$x^{\mu_i} \cdot \left(1 + \sum_{m \geq 1} a_m x^m\right), \quad i = 1, 2, \;\; \mu_1, \mu_2 \in \mathbb{C}, \;\; \mu_1 \neq \mu_2$$

for two distinct complex numbers μ_1 and μ_2. **Solve the differential equation you create.** (Hint: The two numbers μ_1, μ_2 in some of the easier examples are integers.)

The differential equation is ________________________________.

The two linearly independent solutions are ________________________________.

4. (10 pts) Find the general solution to the following differential equation

$$y'' - 5y' + 6y = e^{2x}$$

(Your answer should involve some unspecified constants.)

$y =$ ____.

5. (10 pts) Let C_1 and C_2 be the closed curves

$$C_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \quad C_2 = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 = 36\}$$

on the (x, y)-plane, oriented counterclockwise. Consider the line integrals

$$\oint_{C_i} \frac{(x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y}{x^2 + y^2}, \quad i = 1, 2\,.$$

(a) Are the two integrals $\displaystyle\oint_{C_1} \frac{(x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y}{x^2 + y^2}$ and $\displaystyle\oint_{C_2} \frac{(x - y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y}{x^2 + y^2}$ equal? Why? (**Justify your answer**.)

(b) Evaluate these two line integrals.

$$\oint_{C_1} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\hspace{4cm}}$$

$$\oint_{C_2} \frac{(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y}{x^2+y^2} = \underline{\hspace{4cm}}$$

6. (10 pts) Let $S = \partial D$ be the boundary of the solid region D contained in the cylinder $x^2 + y^2 = 4$ between $z = x$ and $z = 8$, i.e.

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4,\ x \leq z \leq 8\}.$$

Let $\mathbf{n}$ be the unit normal vector field on S pointing outward relative to D. Calculate the flux

$$\iint_S \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}S$$

of the vector field

$$\mathbf{F} = \langle x, y^2, z + y \rangle\ = x\,\vec{i} + y^2\,\vec{j} + (z + y)\,\vec{k}\,.$$

$\iint_S \mathbf{F} \cdot \mathbf{n}\ \mathrm{d}S =$ ________________________.

7. (10 pts) Find the general solution to the following system of differential equations

$$\frac{d}{dt}\mathbf{X} = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} \mathbf{X}, \qquad \text{where } \mathbf{X} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

(Your answer should involve some unspecified constants.)

$\mathbf{X} =$ ____.

8. (10 pts) Find a recursion formula for the coefficients a_n's of a power series expansion of a function

$$y(x) = 1 + \sum_{n=1}^{\infty} a_n \, x^n$$

defined on $(-1, 1)$ which satisfies the following differential equation

$$(x-1)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

Recursion formula: ____.

9. (10 pts) Find the general solution to the system of linear ordinary differential equation

$$\frac{d}{dx}u(x) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} u(x), \quad \text{where} \quad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}$$

(Your answer should involve some unspecified constants.)

$u(x) =$ ____

10. (10 pts) Mark the each of the following following five (5) statements true or false. You do **not** need to justify your answer.

__________ A. Suppose A is a 4×4 matrix and a is a real number such that a, $a+1$, $a+2$ and $2a+3$ are the eigenvalues of A; in other words the characteristic polynomial of A is

$$\det(\lambda \cdot \mathrm{I}_4 - A) = (\lambda - a)(\lambda - a - 1)(\lambda - a - 2)(\lambda - 2a - 3)\,.$$

Then A is diagonalizable.

__________ B. For every 2×2 matrix A of rank 1 there exists a 2×2 matrix B of rank 1 such that $A \cdot B = 0$, where 0 is the 2×2 matrix with all zero entries.

__________ C. The differential equation $x\frac{d^2y}{dx^2} + (\cos(x) - 1)\frac{dy}{dx} = 0$ for the function $y(x)$ on $\mathbb{R}$ is equivalent to a linear ordinary differential equation for $y(x)$ on $\mathbb{R}$ which has *no singular point*.

__________ D. Suppose that $f_0(x)$, $f_1(x)$, and $f_2(x)$ are polynomials in x, and $f_2(x)$ is *not* identically 0. If $y_1(x)$ and $y_2(x)$ are smooth functions (i.e. they can be differentiated infinitely many times) on $\mathbb{R}$ such that $y_1(0) = y_2(0)$, $y_1'(0) = y_2'(0)$, and

$$f_2(x)y_i'' + f_1(x)y_i' + f_0(x)y_i = 0 \quad \text{for } i = 1, 2,$$

then $y_1(x) = y_2(x)$.

__________ E. Suppose that A is a 2×2 matrix with real entries, and $x(t), y(t)$ are two differentiable functions on $\mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{and} \quad x(t)^2 + y(t)^2 = 1$$

for all real numbers $t \in \mathbb{R}$. Then the two eigenvalues λ_1, λ_2 of A has both purely imaginary numbers, i.e. $\lambda_1, \lambda_2 \in \sqrt{-1} \cdot \mathbb{R}$.

Scratch Paper

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