1. Find the inverse of the matrix $A$ and clearly indicate your answer.

$$A := \begin{bmatrix} 1 & 0 & 2 & 0 \\ 5 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 4 & 1 \end{bmatrix}.$$  

Circle the option below which equals the nullity (that is, the dimension of the null space) of $A$.

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) 4  
(f) none of these

Answer: (a) The nullity of any invertible matrix must be zero. $A^{-1} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -5 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & 8 & 1 \end{bmatrix}$.

2. For which value of $k$ will the span of the vectors below be two-dimensional?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ k \end{bmatrix}, \begin{bmatrix} 0 \\ k - 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(a) $-2$  
(b) $-1$  
(c) 0  
(d) 1  
(e) 2  
(f) none of these

Answer: (b) $k = -1$.

3. Consider the $3 \times 3$ matrix

$$A := \begin{bmatrix} 0 & 0 & a - 1 \\ 3 & a & -5 \\ 4 & 0 & -4 \end{bmatrix}.$$  

What value(s) of $a$ make this matrix non-invertible?

(a) $a = 0, 1$  
(b) $a = 0, 2$  
(c) $a = 1, 2$  
(d) $a = \pm \sqrt{2}$  
(e) $a = 2$  
(f) none of these

Answer: (a) $a = 0, 1$.

4. Let $T : V \to V$ be a linear map of a finite dimensional vector space $V$ to itself. The trace of $T$ is defined as the trace of the matrix $\mathfrak{M}^\mathcal{B} (T)$ of $T$ for any basis $\mathcal{B}$ of $V$. It is a true fact that this trace is independent of the choice of the basis $\mathcal{B}$. Let $V$ be the vector space of polynomials of degree at most three, and let $T$ be given by

$$T(p) := x^2 p''' - x^2 p'' - p' + p.$$  

Find the trace of $T$.

(a) $-4$  
(b) $-3$  
(c) $-2$  
(d) $-1$  
(e) 0  
(f) none of these

Answer: (a) $-4$. 
5. In the matrix $A$ from problem 3, let $a = 0$ so that

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix}.$$ 

Which of the following ordered lists of vectors is a cycle of generalized eigenvectors corresponding to eigenvalue $\lambda = -2$?

(a) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, (c) $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$, (e) $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, (f) none of these

Answer: (d).

6. Considering again the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 3 & 0 & -5 \\ 4 & 0 & -4 \end{bmatrix},$$

Find the vector-valued function $X(t)$ satisfying

$$\frac{d}{dt}X(t) = AX(t) \text{ and } X(0) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}.$$ 

(a) $X(t) = \begin{bmatrix} 4(t + 1)e^{-2t} \\ (14t + 4)e^{-2t} \\ 4(2t + 1)e^{-2t} \end{bmatrix}$

(b) $X(t) = \begin{bmatrix} 4(t + 1) \\ (14t + 11) - 7e^{2t} \\ 4(2t + 1) \end{bmatrix}$

(c) $X(t) = \begin{bmatrix} 4(2t + 1) \\ (14t + 11) - 7e^{2t} \\ 4(t + 1) \end{bmatrix}$

(d) $X(t) = \begin{bmatrix} 4(2t + 1) \\ (14t + 11)e^{2t} - 7e^{-t} \\ 4(2t + 1)e^{2t} \end{bmatrix}$

(e) $X(t) = \begin{bmatrix} 4(t + 1) \\ (14t + 11)e^{2t} - 7e^{-t} \\ 4(2t + 1)e^{2t} \end{bmatrix}$

(f) none of these

Answer: (d).

7. Solve the ODE

$$y'' + y' - 6y = 0$$

subject to the initial conditions $y(0) = 0$, $y'(0) = 5$ and clearly indicate your result. Which option below equals $y''(0)$?

(a) 7 (b) 14 (c) 21 (d) 28 (e) 35 (f) none of these

Answer: (e) $y''(0) = 35$. The solution of the IVP is $y(t) = e^{2t} - e^{-3t}$.

8. Solve the ODE

$$\frac{d^2y}{dt^2} + 4 \frac{dy}{dt} + 3y = (4t + 2)e^{-t}$$

subject to the initial conditions $y(0) = -1$, $y'(0) = 1$ and clearly indicate your result. Which option below equals $y(1)$?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2 (f) none of these

Answer: (c) $y(1) = 0$. The solution to the IVP is $y(t) = (t^2 - 1)e^{-t}$.

9. A weight of 1 g is hanging on the end of a spring suspended inside a fluid-filled container from the top. The spring constant of this spring is $k = 4$ dyn cm$^{-1}$ (= 4 g s$^{-2}$), and the fluid exerts 4 dyn of resistive force for every 1 cm s$^{-1}$ of velocity. The spring is displaced 5 cm from its equilibrium position in the upwards/positive direction and given an initial velocity of 11 cm s$^{-1}$ moving towards the equilibrium position. Find the formula for the displacement from equilibrium as a function of time and clearly indicate the result. At what time (if ever) will the weight first reach the equilibrium position?

(a) 2 s (b) 3 s (c) 5 s (d) 7 s (e) 11 s (f) none of these

Answer: (c) 5 s. The height in centimeters as a function of seconds after the start is $x(t) = (5 - t)e^{-2t}$. 
10. Consider the differential equation

\[ y'' + (4x + 2)y' + (4x^2 + 4x + 2)y = 0. \]

The function \( y_1 = e^{-x^2} \) solves this ODE. Use reduction of order to find the general solution. After recording the general solution, solve the IVP \( y(-1) = 1, y'(-1) = 0 \) and circle the option below which agrees with your answer.

(a) \( y = e^{-x^2} \)  
(b) \( y = e^{-(x-1)^2} \)  
(c) \( y = e^{-(x+1)^2} \)  
(d) \( y = e^{x^2} \)  
(e) \( y = e^{(x-1)^2} \)  
(f) \( y = e^{(x+1)^2} \)

Answer: (c) \( y = e^{-(x+1)^2} \). The general solution is \( y = C_1 e^{-x^2} + C_2 e^{-(x+1)^2} \).

11. Let \( A \) equal the matrix:

\[
\begin{pmatrix}
0 & 2 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

Compute \( e^{tA} \) for all \( t \). Which of the following expressions equals the entry of \( e^{tA} \) in row 1 and column 2?

(a) \( e^t \)  
(b) \( \cos t \)  
(c) \( \sin t \)  
(d) \( e^t \cos t + \sin t \)  
(e) \( e^t \cos t - \sin t \)  
(f) none of these

Answer: (d). The matrix exponential is \( e^{tA} = \begin{pmatrix} \cos t & e^t \cos t + \sin t & -\sin t \\ 0 & e^t & 0 \\ \sin t & e^t \cos t - \sin t & \cos t \end{pmatrix} \).

12. The following linear autonomous system has two equilibrium points. Describe the types of these points.

\[
\begin{align*}
\frac{dx}{dt} &= (y^2 - 1) \\
\frac{dy}{dt} &= -y + x
\end{align*}
\]

(a) stable node, center  
(b) saddle point, center  
(c) saddle point, stable spiral  
(d) unstable node, center  
(e) saddle point, unstable spiral  
(f) degenerate node, center

Answer: (c) This system has a stable spiral at \((-1, -1)\) and a saddle point at \((1, 1)\).