FINAL EXAM, MATH 240: CALCULUS III
DECEMBER 17, 2015

No books, paper or electronic device may be used, other than a hand-written notecard at most 4” × 6” in size. Please turn off your cell phones.

Please show all your work. Make sure to write your final answer in the designated location.

NAME (PRINTED):

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RECIPIATION TIME: M 8-9 T 8:30-9:30 W 8-9 Th 8:30-9:30 F 8-9
M 9-10 T 9:30-10:30 W 9-10 Th 9:30-10:30 F 9-10

My signature below certifies that I have complied with the University of Pennsylvania’s code of academic integrity in completing this examination.

Your signature

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1. Decide whether the following statements are true or false. You do not need to show work.

1. A homogeneous system of 7 linear equations in 5 unknowns always has an infinite number of solutions.
   Circle one: True False

2. Every upper triangular matrix is invertible.
   Circle one: True False

3. If $A$ and $B$ are $n \times n$ matrices with eigenvalues $\lambda$ and $\mu$ respectively then $\lambda - \mu$ is an eigenvalue of $A - B$.
   Circle one: True False

4. The set of all solutions to the vector differential equation

   \[ x'(t) = \begin{bmatrix} t & -t & t^2 \\ 1 & -1 & t \\ 1 & 2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix} \]

   is a vector space of dimension 3.
   Circle one: True False
5. Let \( P_1(D) \) and \( P_2(D) \) be polynomial differential operators. If \( y_1 \) is a solution to \( P_1(D)y = 0 \) and \( y_2 \) is a solution to \( P_2(D)y = 0 \), then \( y_1 + y_2 \) is a solution to \( P_1(D)P_2(D)y = 0 \).

   Circle one: True False

6. If \( I_n \) is the \( n \times n \) identity matrix and \( A \) is an \( n \times n \) matrix such that \( A^2 \) is the zero matrix, then \( I_n + A \) is invertible with inverse \( I_n - A \).

   Circle one: True False

7. When using the method of variation of parameters to find the particular solution to a second order differential equation, only one solution to the homogeneous equation is necessary.

   Circle one: True False

8. A polynomial differential operator of order \( n \) annihilates every polynomial of degree \( n - 1 \).

   Circle one: True False

9. If \( A \) is a \( 3 \times 3 \) nondefective matrix whose eigenvalues are \( -2 \pm 3i \) and \( -1 \) then every solution to \( \mathbf{x}' = A\mathbf{x} \) approaches 0 as \( t \) goes to \( \infty \).

   Circle one: True False
2. Compute the inverse of

\[
A = \begin{bmatrix}
-1 & 1 & 3 \\
0 & -1 & 2 \\
-1 & 1 & 2
\end{bmatrix}.
\]

\[A^{-1} = \quad \]
3. Find a basis for the column space of

\[
\begin{bmatrix}
1 & 0 & -1 & 2 \\
2 & 1 & 2 & 3 \\
-1 & -4 & 1 & 2
\end{bmatrix}
\]

Make sure to demonstrate that it is actually a basis.
4. If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and \( \text{det}(A) = -3 \), what is

$$\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$$
5. For each of the following, determine whether or not it is a linear transformation.

(a) $S : P_2 \rightarrow \mathbb{R}$ given by $S(p) = p(1)$ (that is, $S(ax^2 + bx + c) = a1^2 + b1 + c$).

(b) $T : M_2(\mathbb{R}) \rightarrow \mathbb{R}$ given by $T(A) = \det(A)$. 
6. Determine the general solution to $y'' - 5y' + 6y = e^x$. 

\[ y = \]
7. Find the general solution of

\[ y'' - 6y' + 9y = e^{3x} \ln x \]

using the reduction of order method and the fact that \( y_1(x) = e^{3x} \) is a solution to the complementary homogeneous equation. (Hint: remember that \( \int \ln x \, dx = x \ln x - x + C. \))
8. Solve the initial value problem

\[ x'(t) = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \]

\[ x = \]
9. Determine a basis for the space of real valued solutions of \((D^6 - 2D^5 + 5D^4)y = 0\).
10. Characterize the equilibrium point of

\[ x'(t) = \begin{bmatrix} -3 & -6 \\ 1 & -7 \end{bmatrix} x(t) \]

as:

- stable or unstable, and
- node, saddle point, degenerate node, center, or spiral.
Consider the system of differential equations

\[
\begin{bmatrix}
  x'(t) \\
  y'(t)
\end{bmatrix} = \begin{bmatrix}
  -1/t & 0 \\
  t^2 & 1/t
\end{bmatrix} \begin{bmatrix}
  x(t) \\
  y(t)
\end{bmatrix} + \begin{bmatrix}
  2 \\
  -t^3
\end{bmatrix}.
\]

 Knowing that: \( \begin{bmatrix} t \\ t \end{bmatrix} \), \( \begin{bmatrix} t \\ 3t \end{bmatrix} \), and \( \begin{bmatrix} 1/t + t \\ t^2 + t \end{bmatrix} \) are solutions to this system of equations, find the general solution.