Solutions: Math 240 Makeup & Placement Exam

Wednesday, August 26, 2015

No books, paper or any electronic device may be used, other than a hand-written note sheet at most 8.5″ × 11″ in size. Please turn off your cell phones.

This examination consists of twelve (12) long-answer questions. Please show all your work. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers**.

NAME (PRINTED):

INSTRUCTOR:

TA:

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

1	2	3	4	5	6	7	8	9	10	11	12	Total

1. Let A be a 5×5 matrix with $\text{rank}(A) = 3$. Which of the following statements are true?

   a. 0 is an eigenvalue of A.

   b. The first four rows of A are linearly dependent.

   c. A is diagaonalizable.

   d. $\text{rank}(A^2) \geq 1$.

   e. A is defective.

   Answer. a, b, d are true; c, e are false.

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2. Let V be the vector space over $\mathbb{R}$ consisting of all polynomials $p(x, y \in \mathbb{R}[x, y]$ in two variables x and y with real coefficients whose total degree is at most 2. (If $p(x, y) \in V$ is not the zero polynomial, then the $\deg(p(x, y))$ is 0, 1 or 2.)

   (a) Find a basis of V and determine the dimension of V.

   (b) Let $T : V \to V$ be the linear transformation from V to itself, which sends every element $p(x, y) \in V$ to

   $$T(p(x, y)) := y\frac{dp}{dx} + x\frac{dp}{dy}.$$

   Find the matrix representation of this linear transformation T with respect to the basis you picked in (a).

   (c) Find the eigenvalues of this matrix.

   Answer. dim(V)=6. The best basis is $1, x, y, x^2 - y^2, x^2 + y^2, xy$. The transformation is:

$$1 \to 0$$

$$x \to y$$

$$y \to x$$

$$x^2 - y^2 \to 0$$

$$x^2 + y^2 \to 4xy$$

$$xy \to x^2 + y^2.$$

The eigenvalues are therefore $0, \pm 1, \pm 2$.

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3. Let $A = \begin{pmatrix} 2 & 0 & k \\ 0 & 3 & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is a parameter. For which value of the parameter k is the matrix A diagonalizable?

   Answer. For $k \neq 2$.

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4. Let $A = \begin{pmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ Find A^{11}.

   Answer. $\det(\lambda - A) = (\lambda - 1)^3$, $A^{11} = -10\,\mathrm{I}_3 + 11\,A = \begin{pmatrix} -10 & 22 & -11 \\ -11 & 23 & -11 \\ -11 & 22 & -10 \end{pmatrix}$.

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5. Solve the initial-value problem

$$y'' + 3y' = 6t, \qquad y(0) = 0, \quad y'(0) = 0$$

   for $y(t)$.

   Answer. General solution: $t^2 - \frac{2}{3}t + c_1 + c_2 e^{-3t}$.

   Solution of the initial value problem: $t^2 - \frac{2}{3}t + \frac{2}{9} - \frac{2}{9}e^{-3t}$.

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6. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 4\,e^{-2t}\ln t, \qquad t > 0$$

   Answer. A particular solution is $y_p(t) = t^2 e^{-2t}(2\ln t - 3)$. (cf. the textbook, example 6.7.3)

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7. Let $B := \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. Find the general solution of the system of first order linear differential equations

$$\frac{d}{dt}\mathbf{x}(t) = B \cdot \mathbf{x}(t),$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Answer:

$$x = (a + (2b - c)t + 2ct^2)e^{-t}$$

$$y = (b + 2ct)e^{-t}$$

$$z = ce^{-t}.$$

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8. Find a (particular) solution $y_p(x)$ of the differential equation

$$\left(\frac{d}{dx} + 1\right)^3 \left(\frac{d}{dx} - 1\right) y(x) = -240\, x^2 e^{-x} + 120\, e^{-x}.$$

Answer. $y_p(x) = 2x^5 e^{-x} + 5x^4 e^{-x}$.

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9. Let $A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -9 & 11 \end{pmatrix}$. Compute the matrix exponential e^{tA}.

Answer: The upper left 3x3 block is: $e^{3t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$.

The lower right 2x2 block is: $e^{5t} \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} + e^{2t} \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$.

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10. Consider the following differential equation with a real parameter k

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + 8ky = \cos(2t) \tag{1}$$

Determine all values of the parameter $k \in \mathbb{R}$ such that every solution $y(t)$ of the differential equation (1) is bounded as $t \to \infty$ (i.e. there exists a constant C depending on the solution $y(t)$ such that $|y(t)| \leq C$ for all $t \geq 0$).M

Answer. $\{k \in \mathbb{R} : 0 < k < 8\}$.

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11. Find all equilibrium points of the differential equation

$$\frac{dx}{dt} = y(t), \quad \frac{dy}{dt} = -4\sin(x(t)) - \frac{y(t)}{25}$$

and determine/classify each of the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc.

Answer: $y = 0, \quad x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$: stable spiral.

$y = 0, \quad x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$: saddle point.

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12. (a) Find the general solution to the system of differential equations:

$$x' = 2x + 3y, \qquad y' = x + 4y$$

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions

$$x(0) = 0, \quad y(0) = 2.$$

Answer:

(a) $c_1 e^t \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $c_1 = -\frac{1}{2}, c_2 = \frac{3}{2}$
