1) True False
   1. True.
   2. False.
   3. True.
   4. False.
   5. False.
   6. False.
   7. False.
   8. True.

2) Short Answer
   1. 12.
   2. $2\pi i$.
   3. $e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, e^{i\frac{7\pi}{2}}$.
   4. Essential.
   5. 0. The function is analytic.
   6. 7. All the Fourier coefficients are zero except for 7, the coefficient on $\cos 2x$.
   7. $\sqrt{2}$.
   8. 3.
1. The integrand has simple poles at \(z = i, -i\). Both poles are inside the contour \(C\).

\[
\text{Res}\left(\frac{z^2 - 4}{z^2 + 1}, i\right) = \frac{i^2 - 4}{2i} = -\frac{5}{2i} = \frac{5i}{2}.
\]

\[
\text{Res}\left(\frac{z^2 - 4}{z^2 + 1}, -i\right) = \frac{(-i)^2 - 4}{-2i} = -\frac{5}{-2i} = \frac{5i}{2}.
\]

So

\[
\int_C \frac{z^2 - 4}{z^2 + 1} \, dz = 2\pi i \left(\frac{5i}{2} - \frac{5i}{2}\right) = 0.
\]

2. The integrand has a pole of order 2 at 2. The pole is inside the contour \(C\).

\[
\text{Res}\left(\frac{e^z}{(z - 2)^2}, 2\right) = \lim_{z \to 2} \frac{d}{dz} \left[\frac{e^z}{(z - 2)^2}\right] = \lim_{z \to 2} e^z = e^2.
\]

So

\[
\int_C \frac{e^z}{(z - 2)^2} \, dz = 2\pi i (e^2) = 2e^2\pi i.
\]

3. The integrand, which we think of as \(\frac{\sin \frac{z}{z \cos z}}{z \cos z}\), has simple poles at 0 and all odd multiples of \(\frac{\pi}{2}\). The poles inside \(C\) are \(\frac{\pi}{2}, \frac{3\pi}{2}\), and \(\frac{5\pi}{2}\). Note that \(z = 0\) is a removable singularity and hence its residue is 0.

\[
\text{Res}\left(\frac{\sin z}{z \cos z}, \frac{\pi}{2}\right) = \frac{1}{0 - \left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}
\]

\[
\text{Res}\left(\frac{\sin z}{z \cos z}, \frac{3\pi}{2}\right) = \frac{-1}{0 - \left(\frac{3\pi}{2}\right)(-1)} = -\frac{2}{3\pi}
\]

Similarly \(\text{Res}\left(\frac{\sin z}{z \cos z}, \frac{3\pi}{2}\right) = -\frac{2}{5\pi}\). So

\[
\int_C \frac{\tan \frac{z}{z}}{z} \, dz = -2\pi i \left(\frac{2}{\pi} + \frac{2}{3\pi} + \frac{2}{5\pi}\right) = -2\pi i \left(\frac{30 + 10 + 6}{15\pi}\right) = -\frac{92}{15} i.
\]
4)

1. \( f(z) \) has three different Laurent expansions about \( z = 1 \). One good in \(|z - 1| < 1\) one good in \(1 < |z - 1| < 4\) and one good in \(|z - 1| > 4\).

2. Using partial fractions you find

\[
f(z) = -\frac{1}{z + 3} + \frac{1}{z - 2}
\]

Recall \( \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \) good for \(|w| < 1\). So

\[
\frac{1}{4 + (z - 1)} = \frac{1}{4(1 + \frac{z - 1}{4})} = \frac{1}{4} \sum_{n=0}^{\infty} (-\frac{1}{4})^n (z-1)^n = -\sum_{n=0}^{\infty} (-\frac{1}{4})^{n+1} (z-1)^n
\]

Similarly \( \frac{1}{1-w} = \sum_{n=0}^{\infty} - (\frac{1}{w})^{n+1} \) for \(|w| > 1\). So

\[
\frac{1}{-1 + (z - 1)} = \frac{1}{-(1 - (z - 1))} = -\sum_{n=0}^{\infty} -(z-1)^{-(n+1)} = \sum_{n=0}^{\infty} (z-1)^{-(n+1)}
\]

Thus

Answer:

\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{5} (-\frac{1}{4})^{n+1} (z-1)^n + \sum_{n=0}^{\infty} \frac{1}{5} (z-1)^{-(n+1)}
\]

5)

\[
b_n = \frac{2}{3} \int_{0}^{3} (3x - 4) \sin \frac{n\pi}{3} x \, dx
\]

\[
= \frac{2}{3} \left( -\frac{3}{n\pi} (3x - 4) \cos \frac{n\pi}{3} x + \frac{9}{n^2\pi^2} 3 \sin \frac{n\pi}{3} x \right) \bigg|_{0}^{3}
\]

\[
= \frac{2}{3} (-\frac{15}{n\pi} (-1)^n + 0 + \frac{-12}{n\pi} - 0) = -\frac{2}{3n\pi} (12 + (-1)^n 15)
\]

So the Fourier sine expansion is

Answer:

\[
\sum_{n=1}^{\infty} -\frac{2}{3n\pi} (12 + (-1)^n 15) \sin \frac{n\pi}{3} x
\]
6) 1. Make the substitution \( z = e^{ix} \) so \( dx = \frac{1}{iz} dz \) and \( \cos x = \frac{1}{2}(z + \frac{1}{z}) \). Thus if \( C \) is the unit circle our integral becomes

\[
\int_C \frac{1}{2 - \frac{1}{2}(z + \frac{1}{z})} \frac{1}{iz} dz = 2i \int_C \frac{1}{z^2 - 4z + 1} dz.
\]

The integrand has simple poles at \( z = 2 \pm \sqrt{3} \), but only \( 2 - \sqrt{3} \) is inside the contour \( C \). Thus

\[
\int_C \frac{1}{z^2 - 4z + 1} dz = 2\pi i (\text{Res}(\frac{1}{z^2 - 4z + 1}, 2 - \sqrt{3})) = 2\pi i \frac{1}{2(2 - \sqrt{3}) - 4} = -\frac{\pi i}{\sqrt{3}}.
\]

Thus

\[
\text{Answer: } 2i(-\frac{\pi i}{\sqrt{3}}) = \frac{2\pi}{\sqrt{3}}.
\]

2. For this type of integral we compute \( \int_{-\infty}^{\infty} \frac{x^2 e^{(i\pi x)}}{(x^2+1)(x^2+2)} dx \) and take the real part. So

\[
\lim_{R \to \infty} \int_{-R}^{R} \frac{x^2 e^{(i\pi x)}}{(x^2+1)(x^2+2)} dx = \lim_{R \to \infty} \int_{I_R} \frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)} dz = \lim_{R \to \infty} \left( \int_{C_R} + \int_{S_R} \right) \frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)} dz.
\]

Where \( I_R = \{-R \leq x \leq R\} \), \( S_R \) is the circle of radius \( R \) about the origin and \( C_R \) is their union. The only singularities inside \( C_R \) are two poles at \( i \) and \( \sqrt{2}i \). so

\[
\int_{C_R} \frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)} dz = 2\pi i (\text{Res}(\frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)}, i) + \text{Res}(\frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)}, \sqrt{2}i))
\]

\[
= 2\pi i (-\frac{e^{-\pi}}{2i} + \frac{-2e^{-\pi\sqrt{2}}}{-2\sqrt{2}i}) = -\frac{\pi(2e^{-\pi\sqrt{2}} + \sqrt{2}e^{-\pi})}{\sqrt{2}}.
\]

Also

\[
\lim_{R \to \infty} \int_{S_R} \frac{z^2 e^{(i\pi z)}}{(z^2+1)(z^2+2)} dz = 0.
\]

by a theorem from class. Thus

\[
\text{Answer: } -\frac{\pi(2e^{-\pi\sqrt{2}} + \sqrt{2}e^{-\pi})}{\sqrt{2}}.
\]
7) Take the Laplace transform of both sides to get

$$\mathcal{L}\{y'' - 5y' + 4y = 0\} = 0$$

so

$$s^2 \hat{y} - sy(0) - y'(0) - 5s\hat{y} + 5y(0) + 4\hat{y} = 0$$

$$\hat{y} = \frac{2}{s^2 - 5s + 4} = \frac{2}{3} \frac{1}{s - 4} - \frac{2}{3} \frac{1}{s - 1}.$$ 

Thus

$$y(t) = \mathcal{L}^{-1}\{\hat{y}\} = \frac{2}{3} \mathcal{L}^{-1}\{\frac{1}{s - 4}\} - \frac{2}{3} \mathcal{L}^{-1}\{\frac{1}{s - 1}\}.$$

So

Answer:

$$y(t) = \frac{2}{3} e^t - \frac{2}{3} e^t.$$ 

8) 1. Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function. The real part of \( f \) is \( u(x, y) \).

\[
    u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{xy} - v_{yx} = 0,
\]

where the second equality follows from the Cauchy Riemann equations.

2. The function \( u(x, y) = xy - 1 \) is clearly harmonic. If \( v(x, y) \) is its harmonic conjugate then \( v_y = u_x = y \) and \( v_x = -u_y = -x \). Integrating the first equation with respect to \( y \) gives

\[
    v(x, y) = \frac{1}{2} y^2 + g(x).
\]

Thus \( v_x = g'(x) \) but this must be \(-x\) so \( g(x) = -\frac{1}{2} x^2 + c \). So

Answer:

$$v(x, y) = \frac{1}{2} (y^2 - x^2 + c)$$

9) The equation becomes

$$X''T - 2XT'' + 7XT' + XY = 0.$$ 

Dividing by \( XT \) and rearranging we get

$$\frac{X''}{X} = \frac{1}{T} (2T'' - 7T') - 1.$$
Since each side is a function of a different variable they must both be constant. Let the constant be \( k \). So we get equations for \( X \) and \( T \):

\[
X'' - kX = 0
\]

\[
2T'' - 7T' - (k + 1)T = 0
\]

The boundary conditions for \( u \) imply \( X(0) = 0 \) and \( X(7) = 0 \). As we have seen many times this leads to solutions

\[
X_n(x) = \sin\left(\frac{n\pi}{7}x\right).
\]

Plugging into the equation for \( T \) give

\[
T_n(x) = a_n e^{\lambda_n t} + b_n e^{\lambda'_n t}
\]

where \( \lambda_n = \frac{7 + \sqrt{49 + 8(1 - \frac{4n^2}{49})}}{4} \) and \( \lambda'_n = \frac{7 - \sqrt{49 + 8(1 - \frac{4n^2}{49})}}{4} \). Thus the solution to the PDE that can be written as the product to \( X(x) \) and \( T(t) \) are

**Answer:**

\[
u A + \nu B = 0 \quad \text{and} \quad A\nu e^{3\nu} - B\nu e^{-3\nu} = 0.
\]

One may easily check that the only \( A \) and \( B \) satisfying this are \( A = 0 = B \). Thus we only get the trivial solution again.

Now if \( \lambda > 0 \), so we can write it \( \lambda = \nu^2 \) then out solutions to the ODE are \( y(x) = A e^{\nu x} + B e^{-\nu x} \). Plugging in the boundary conditions we see

\[
u A + \nu B = 0 \quad \text{and} \quad A\nu e^{3\nu} - B\nu e^{-3\nu} = 0.
\]

One may easily check that the only \( A \) and \( B \) satisfying this are \( A = 0 = B \). Thus we only get the trivial solution again.

Now if \( \lambda > 0 \), so we can write it \( \lambda = \nu^2 \) then out solutions to the ODE are \( y(x) = A \cos \nu x + B \sin \nu x \). Plugging in the boundary conditions we see \( B\nu = 0 \) so \( B = 0 \) and \( A\nu \sin 3\nu = 0 \). To get non trivial solutions we need \( 3\nu = n\pi \). To get all eigenfunctions we just need to consider non-negative integers \( n \). (Note: when \( n = 0 \) we get the eigen function \( y = 1 \) with eigenvalue \( \lambda = 0 \).) Thus we have
Answer: Eigenvalues $\lambda_n = (\frac{n\pi}{3})^2$ with eigenfunctions $\cos \frac{n\pi}{3}x$, where $n$ runs through all non-negative integers.

11) We are looking for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$. So $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$. Thus the equation is

$$y'' + 2y' + y = \sum_{n=0}^{\infty} [(n + 2)(n + 1)a_{n+2} + 2(n + 1)a_{n+1} + a_n]x^n = 0.$$

So

Answer:

$$a_{n+2} = \frac{-a_n - (n + 1)a_{n+1}}{(n + 2)(n + 1)}.$$